

FOURIER LAW, PHASE TRANSITIONS AND THE STATIONARY STEFAN PROBLEM

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ABSTRACT. We study the one-dimensional stationary solutions of an integro-differential equation derived by Giacomini and Lebowitz from Kawasaki dynamics in Ising systems with Kac potentials, [5]. We construct stationary solutions with non zero current and prove the validity of the Fourier law in the thermodynamic limit showing that below the critical temperature the limit equilibrium profile has a discontinuity (which defines the position of the interface) and satisfies a stationary free boundary Stefan problem. Under-cooling and over-heating effects are also studied. We show that if metastable values are imposed at the boundaries then the mesoscopic stationary profile is no longer monotone and therefore the Fourier law is not satisfied. It regains however its validity in the thermodynamic limit where the limit profile is again monotone away from the interface.

1. INTRODUCTION

When hydrodynamic or thermodynamic limits are performed in systems which are in the phase transitions regime we may observe perfectly smooth profiles develop singularities with the appearance of sharp interfaces. We shall study the phenomenon in stationary non equilibrium states which carry non zero steady currents, the general context is the one where the Fourier law applies, but here it is complemented by a free boundary problem due to the presence of interfaces. We work at the mesoscopic level considering a model which has been derived in [5] from Ising systems with Kac potentials and Kawasaki dynamics, and derive in the hydrodynamic limit macroscopic profiles with an interface which satisfy a stationary Stefan problem and obey the Fourier law.

The mesoscopic model is defined in terms of a free energy functional, the Lebowitz and Penrose [L-P] functional (see (2.1) in the next section) which is a non local version of the scalar Ginzburg-Landau (or Allen-Cahn or Cahn-Hilliard) functional. Its thermodynamic free energy density is obtained by minimizing the L-P functional over profiles with fixed total magnetization density and then taking the thermodynamic limit where the spatial size of the system diverges. It is found that the phase diagram (of free energy density versus magnetization density) obtained in this way has a non trivial flat interval $[-m_\beta, m_\beta]$ (indicative of a phase transition) when the inverse temperature β is above the critical value (equal to 1 here). This is in qualitative and quantitative agreement with the thermodynamics of the underlying Ising model with Kac potentials, see Chapter 9 in [9] and references therein. The axiomatic theory for such phase diagrams predicts that the values inside $(-m_\beta, m_\beta)$ do not appear in any stationary local equilibrium state, so that a macroscopic magnetization density profile will have a discontinuity if it assumes values both smaller than $-m_\beta$ and larger than m_β .

This is just what we see. We fix $\beta > 1$ and study the stationary solutions of the equations of motion $\frac{dm}{dt} = -\operatorname{div} I$, i.e. $\operatorname{div} I = 0$, I the local current (of the conserved order parameter, the magnetization density m here). By a gradient flow assumption on its constitutive law, I is supposed proportional to the gradient of the functional derivative of the L-P functional: due to the non local structure of the latter, $\frac{dm}{dt} = -\operatorname{div} I$ is an integro-differential equation (see (2.19) in the next section), which is the same as the one derived by Giacomin and Lebowitz from the Ising system, [5], and which has been much studied in the past years, [7], [1], [6]. We look for solutions of $\operatorname{div} I = 0$ with a planar symmetry thus reducing to a one dimensional problem and prove existence and smoothness of solutions with a steady non-zero current. However in the hydrodynamic limit where the size L of the system diverges, the stationary profile, once expressed in macroscopic space units (i.e. proportional to L), is proved to converge to a discontinuous limit profile, solution of a stationary free boundary problem, the stationary Stefan problem, in agreement with the axiomatic macroscopic theory. The mesoscopic theory is in this respect in complete agreement with the macroscopic one, the mesoscopic profiles are smooth versions of the macroscopic ones, they are monotone as well and the current is proportional to [minus] the magnetization density gradient in agreement with the Fourier law which we may then say to be valid at the mesoscopic level as well.

The mesoscopic theory has however a richer and more complex structure even in the macroscopic limit. This is seen for instance if we impose boundary conditions which force metastable values at the boundaries, the metastable region being made of two separate intervals called the plus and the minus metastable phases (according to the sign of the magnetization) which (together with the spinodal region) are contained in the “forbidden region” $(-m_\beta, m_\beta)$. With boundary conditions one in the minus, the other in the plus metastable phases the mesoscopic stationary magnetization density profiles are not monotone anymore. We have the “paradoxical” result of a positive [magnetization] current when also the total magnetization gradient is positive having fixed at the left and right respectively a negative and a positive metastable value of the magnetization. The mesoscopic stationary profile is then first decreasing, then increasing and then again decreasing. The Fourier’s law is therefore not satisfied but, in the thermodynamic limit, the region where the profile increases shrinks to a point, which is where the limit profile has a discontinuity (a sharp interface). Elsewhere the profile is always decreasing in agreement with the Fourier’s law (as the current is positive). The stationary profile has therefore values all in the metastable region (except at the interface which macroscopically is only a point). All the issues presented in this introduction are discussed in some more details in the next section, proofs are given in the remaining ones.

2. MODEL, BACKGROUNDS AND MAIN RESULTS

The free energy functional to which we have been referring so far is defined on functions $m \in L^\infty(\Lambda, [-1, 1])$, Λ a bounded measurable subset of \mathbb{R}^d , as

$$\begin{aligned} F_{\beta, \Lambda}(m|m_{\Lambda^c}) &= F_{\beta, \Lambda}(m) + \frac{1}{2} \int_{\Lambda} \int_{\Lambda^c} J(x, y)[m(x) - m_{\Lambda^c}(y)]^2 \\ F_{\beta, \Lambda}(m) &= \int_{\Lambda} \phi_{\beta}(m) + \frac{1}{4} \int_{\Lambda} \int_{\Lambda} J(x, y)[m(x) - m(y)]^2 \end{aligned} \quad (2.1)$$

where $J(x, y) = J(|x - y|)$ is a smooth, translational invariant, probability kernel of range 1; $m_{\Lambda^c} \in L^\infty(\Lambda^c, [-1, 1])$ is a fixed external profile and

$$\phi_{\beta}(m) = -\frac{1}{2}m^2 - \frac{1}{\beta}S(m), \quad -S(m) = \frac{1+m}{2} \log\left(\frac{1+m}{2}\right) + \frac{1-m}{2} \log\left(\frac{1-m}{2}\right) \quad (2.2)$$

To simplify the analysis we suppose Λ a cube and consider Neumann boundary conditions, namely the functional

$$F_{\beta, \Lambda}^{\text{neum}}(m) = \int_{\Lambda} \phi_{\beta}(m) + \frac{1}{4} \int_{\Lambda} \int_{\Lambda} J^{\text{neum}}(x, y)[m(x) - m(y)]^2 \quad (2.3)$$

where $J^{\text{neum}}(x, y) = \sum_{z \in R_{\Lambda}(y)} J(x, z)$ with $R_{\Lambda}(y)$ the set image of y under reflections

of the cube Λ around its faces. In $d = 1$, if $\Lambda = \epsilon^{-1}[-1, 1]$, $J^{\text{neum}}(x, y) = J(x, y) + J(x, 2\epsilon^{-1} - y) + J(x, -2\epsilon^{-1} - y)$ ($\epsilon > 0$ is a scaling parameter which will vanish in the thermodynamic limit). With minor modification what follows in the next item “Equilibrium thermodynamics” holds as well for general boundary conditions as those considered in (2.1).

Equilibrium thermodynamics of the mesoscopic model.

(The statements in this paragraph are proved in Section 6.1 of [9]). The thermodynamic free energy density $a_{\beta}(s)$, $s \in [-1, 1]$, is defined as

$$a_{\beta}(s) := \lim_{\Lambda \rightarrow \mathbb{R}^d} \inf \left\{ F_{\beta, \Lambda}^{\text{neum}}(m) \mid \int_{\Lambda} m = s \right\} \quad (2.4)$$

The limit on the r.h.s. indeed exists and it is equal to:

$$a_{\beta} = \phi_{\beta}^* = \text{convex envelope of } \phi_{\beta}(\cdot) \quad (2.5)$$

$\phi_{\beta}^* \equiv \phi_{\beta}$ when $\beta \leq 1$ and $\phi_{\beta}^* \neq \phi_{\beta}$ when $\beta > 1$. More precisely let m_{β} be the positive solution of

$$m_{\beta} = \tanh\{\beta m_{\beta}\}, \quad \beta > 1 \quad (2.6)$$

then $\phi_{\beta}^*(s)$, $s \in (-m_{\beta}, m_{\beta})$, is constant and strictly smaller than $\phi_{\beta}(s)$, while $\phi_{\beta}^*(s) = \phi_{\beta}(s)$ elsewhere. The values of the magnetization in the interval $(-m_{\beta}, m_{\beta})$ are “forbidden”. This is best seen working in the grand canonical ensemble (in other words, using Lagrange multipliers). To this end we add a constant magnetic field h so that the free energy functional becomes

$$F_{\beta, h, \Lambda}^{\text{neum}}(m) = F_{\beta, \Lambda}^{\text{neum}}(m) - h \int_{\Lambda} m \quad (2.7)$$

The grand canonical thermodynamic pressure $p_\beta(h)$ is defined by a minimization problem without constraints:

$$p_\beta(h) = \lim_{\Lambda \rightarrow \mathbb{R}^d} \sup \left\{ -F_{\beta,h,\Lambda}^{\text{neum}}(m) \mid m \in L^\infty(\Lambda, [-1, 1]) \right\} \quad (2.8)$$

Existence of the limit is again a fact and the two thermodynamics defined by the free energy a_β and by the pressure p_β are equivalent, a property called in statistical mechanics “equivalence of ensembles”. Namely p_β and a_β are inter-related as in thermodynamics being one the Legendre transform of the other:

$$p_\beta(h) = \sup \{hs - a_\beta(s) \mid s \in [-1, 1]\}, \quad a_\beta(s) = \sup \{hs - p_\beta(h) \mid h \in \mathbb{R}\} \quad (2.9)$$

For any $\beta > 1$ and any $h \in \mathbb{R}$ any maximizer of (2.8) at least for Λ large enough is a constant function equal to $m_{\beta,h}$ where $m_{\beta,h}$ is the solution of the mean field equation

$$m_{\beta,h} = \tanh\{\beta(m_{\beta,h} + h)\} \quad (2.10)$$

which minimizes $\phi_\beta(s) - hs$ and therefore it is not in $(-m_\beta, m_\beta)$, the values in $(-m_\beta, m_\beta)$ “are therefore forbidden”.

Gibbsian equilibrium thermodynamics.

The thermodynamics obtained above is in qualitative and quantitative agreement with the thermodynamics of the underlying microscopic model, i.e. the Ising system with Kac potential. The Gibbs canonical equilibrium free energy $f_{\beta,\gamma}(m)$ is defined as

$$f_{\beta,\gamma}(m) := \lim_{\delta \rightarrow 0} \lim_{\Lambda_n \rightarrow \mathbb{Z}^d} \frac{-1}{\beta|\Lambda_n|} \log Z_{\Lambda_n,\beta,\gamma} \quad (2.11)$$

$$Z_{\Lambda_n,\beta,\gamma} = \sum_{\sigma_{\Lambda_n} \in \{-1,1\}^{\Lambda_n}} \mathbf{1} \left(\left| \sum_{x \in \Lambda_n} (\sigma_{\Lambda_n}(x) - m) \right| \leq \delta |\Lambda_n| \right) e^{-\beta H_{\gamma,\Lambda_n}(\sigma_{\Lambda_n})}$$

where Λ_n is a sequence of increasing cubes and

$$H_{\gamma,\Lambda}(\sigma_\Lambda) = -\frac{1}{2} \sum_{x \neq y \in \Lambda} J_\gamma(x, y) \sigma_\Lambda(x) \sigma_\Lambda(y), \quad J_\gamma(x, y) = \gamma^d J_\gamma(\gamma|x - y|) \quad (2.12)$$

(Same free energy is obtained for more general regions and boundary conditions). As discussed in Chapter 9 of [9] in $d \geq 2$ for any $\beta > 1$ and $\gamma > 0$ small enough, $f_{\beta,\gamma}(m)$ is flat in an interval $[-m_{\beta,\gamma}, m_{\beta,\gamma}]$ and $m_{\beta,\gamma} \rightarrow m_\beta$ as $\gamma \rightarrow 0$. The original result has been proved in [3] and [2] while the fact that in any $d \geq 1$, $\lim_{\gamma \rightarrow 0} f_{\beta,\gamma}(m) = a_\beta(m)$ is much older and proved by Lebowitz and Penrose, [8].

Axiomatic non equilibrium macroscopic theory.

The basic postulates are (i)–(iv).

(i) *local equilibrium and barometric formula.* The free energy of a macroscopic profile m in the macroscopic (bounded) region $\Omega \subset \mathbb{R}^d$ is given by the local functional:

$$F_{\beta,\Omega}^{\text{macro}}(m) := \int_\Omega a_\beta(m), \quad m \in L^\infty(\Omega, [-1, 1]) \quad (2.13)$$

(ii) *gradient dynamics.* The evolution equation in the interior of Ω is the conservation law (D below denoting functional derivative)

$$\frac{dm}{dt} = -\nabla j, \quad j = -\chi \nabla D F_{\beta, \Omega}^{\text{macro}} = -\chi \nabla a'_{\beta}, \quad a'_{\beta}(s) := \frac{da_{\beta}(s)}{ds} \quad (2.14)$$

(iii) *mobility coefficient.* χ is a mobility coefficient which depends on the dynamical characteristics of the system, we take

$$\chi(s) = \beta(1 - s^2) \quad (2.15)$$

as this is what found when deriving (2.14) from the Ising spins, [5]-[7].

The usual setup for Fourier law has Ω a parallelepiped with different values of the order parameter imposed on its right and left faces and Neumann (or periodic) conditions on the other ones. By the planar symmetry the problem becomes one dimensional and from now on we shall restrict to $d = 1$ taking $\Omega = [-\ell, \ell]$. The stationary profiles $m(x)$, $x \in (-\ell, \ell)$, verify

$$D_{\beta} \frac{dm}{dx} = -j = \text{constant}, \quad D_{\beta}(m) = \chi(m) a''_{\beta}(m) \quad (2.16)$$

and are determined for instance by Dirichlet boundary conditions at $\pm\ell$, namely $m(x) \rightarrow m_{\pm}$ as $x \rightarrow \pm\ell$. To have an increasing profile we shall suppose that $-1 < m_- < -m_{\beta}$ and $1 > m_+ > m_{\beta}$, the opposite case being recovered by symmetry. When $\beta < 1$ the above is well posed as $a''_{\beta} > 0$ but if $\beta \geq 1$ the denominator vanishes. The macroscopic theory then needs a further postulate:

(iv) *The stationary Stefan problem.* There are $x_0 \in (-\ell, \ell)$ and $j < 0$ so that there is a solution $m(x)$ of (2.16) in $(-\ell, x_0)$ with boundary values m_- and $-m_{\beta}$ and in (x_0, ℓ) with boundary values m_{β} and m_+ . The current $-\chi(m(x)) \frac{d}{dx} a'_{\beta}(m(x))$ being equal to j is constant through the interface x_0 : conservation of mass would otherwise impose a motion of the interface against the assumption that the profile is stationary. Observe also that since $a'_{\beta}(m)$ is an increasing function of m in $[-m_{\beta}, m_{\beta}]^c$ and since $h(x)$ is increasing (if $j < 0$) then $m(x)$ is also increasing.

A different formulation of the problem is however more convenient for our purposes. We start by a change of variables, going from m to h . There is a one to one correspondence between m and h when $\{m \geq m_{\beta}\}$ and $\{h \geq 0\}$ and also when $\{m \leq -m_{\beta}\}$ and $\{h \leq 0\}$. The correspondence is given in one direction by (2.10), and in the other by $h = a'_{\beta}(m)$. Expressed in terms of the magnetic field, (2.16) becomes

$$h(x) = \int_{x_0}^x \frac{-j}{\chi(m)}, \quad m = (a'_{\beta})^{-1}(h) \quad (2.17)$$

namely m is regarded as a function of h obtained by inverting $h = a'_{\beta}(m)$ and $\chi(m) = \chi(m(h))$ becomes a function of h as well. (2.17) is then an integral equation in $h(\cdot)$ where however x_0 and j are also unknown: they must be determined by imposing the boundary conditions $h(\pm\ell) = h_{\pm} := a'_{\beta}(m_{\pm})$. All this suggests a new formulation (alternative to the Dirichlet problem) where we assign x_0 and j instead of m_{\pm} . In this way the Stefan problem is written in a compact way as in (2.17) above which is now a “pure” integral equation for $h(\cdot)$ with x_0 and j known data. We shall mostly use in the sequel this latter formulation when proving that the Stefan problem with assigned x_0 and j can be derived from the mesoscopic theory.

As a difference with the Dirichlet problem, in the “ x_0, j problem” there is no “global existence theorem”, in the sense that given x_0 and j there are no solutions if ℓ is too large. Indeed (2.17) with $x_0 = 0$ and $j < 0$ has a “maximal solution” $(h_j(x), m_j(x))$. Namely there is a bounded interval $(-\ell_j, \ell_j)$ such that

$$\lim_{x \rightarrow \pm \ell_j} m_j(x) = \pm 1, \quad \lim_{x \rightarrow \pm \ell_j} h_j(x) = \pm \infty \quad (2.18)$$

(2.17) has no solution if $\ell > \ell_j$ while any other solution of (2.17) with the same j is obtained, modulo translations, by restricting the maximal solution to a suitable interval contained in $(-\ell_j, \ell_j)$. The value ℓ_j is strictly finite because the solution $m(h)$ of $m = \tanh\{\beta h + \beta m\}$ when $h \rightarrow \infty$ and $m(h) \rightarrow 1$ is to first order given by $\frac{dm}{dh} \approx \beta(1 - m^2)$. Thus $\frac{dm}{dx} \approx -j$ in (2.16) when $m \approx 1$ hence $m(\cdot)$ converges to 1 linearly with slope $-j$ (recall $j < 0$). The collection of all the maximal solutions $(h_j(x), m_j(x))$ when $j \in \mathbb{R} \setminus \{0\}$ determines in the sense explained above all the possible solutions of (2.17). Since $\ell_j \rightarrow 0$ as $j \rightarrow \infty$ and $\ell_j \rightarrow \infty$ as $j \rightarrow 0$ it then follows that for any ℓ the Dirichlet problem with data m_{\pm} at $\pm \ell$ ($m_+ \neq m_-$, m_{\pm} in the complement of $[-m_{\beta}, m_{\beta}]$) can be obtained as described above from the collection of all the maximal solutions. By taking limits we can also include m_{β} and $-m_{\beta}$.

By restricting to intervals strictly contained in the maximal interval $[-\ell_j, \ell_j]$ the solution (h, m) of (2.17) is smooth, $\|m\| < 1$, $\chi(m)$ bounded away from 0 and $\|h\| < \infty$. These are the properties of the macroscopic solution which will be repeatedly used in the sequel.

Stationary mesoscopic profiles.

Dynamics is defined using the same postulate of the macroscopic theory, namely it is the gradient flow of the free energy functional which, in the mesoscopic theory is (2.3) (supposing again Neumann conditions). The gradient flow is (D below denoting functional derivative)

$$\begin{aligned} \frac{dm}{dt} &= -\nabla I, \quad I = -\chi \nabla (DF_{\beta, \Lambda}) \\ I &= -\chi \nabla \left(\frac{1}{2\beta} \log \frac{1+m}{1-m} - \int J^{\text{neum}}(x, y) m(y) dy \right) \end{aligned} \quad (2.19)$$

With the choice $\chi = \beta(1 - m^2)$ (that we adopt hereafter) (2.19) becomes the one found in [5] from the Ising spins. We suppose again a planar symmetry to reduce to one dimension, take $\Lambda = \epsilon^{-1}[-\ell, \ell]$ interpreting ϵ^{-1} as the ratio of macroscopic and mesoscopic lengths so that (2.19) becomes

$$\frac{dm}{dt} = -\frac{d}{dx} \left(-\frac{dm}{dx} + \beta(1 - m^2) \frac{d}{dx} J^{\text{neum}} * m \right) \quad (2.20)$$

As in the macroscopic theory it is now convenient to change variables. Define $h(x)$ as

$$h := \frac{1}{2\beta} \log \frac{1+m}{1-m} - J^{\text{neum}} * m \quad (2.21)$$

Then the current I in (2.19) has the expression

$$I = -\chi(m) \frac{dh}{dx}, \quad m = \tanh\{\beta J^{\text{neum}} * m + \beta h\} \quad (2.22)$$

The stationary problem in the x_0, j formulation is then the following. Given any $x_0 \in (-\ell, \ell)$ and $j < 0$, find m and h so that

$$m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}, \quad h(x) = \int_{\epsilon^{-1}x_0}^x \frac{-\epsilon j}{\chi(m)} \quad (2.23)$$

We first consider the simpler antisymmetric case where m and h are both odd functions.

Theorem 2.1. *Let $j \neq 0$, $x_0 = 0$, $\ell > 0$ and smaller than ℓ_j (see (2.18)). Then for any $\epsilon > 0$ small enough there is an antisymmetric pair $(h_\epsilon(x), m_\epsilon(x))$ which solves (2.23) in $\epsilon^{-1}(-\ell, \ell)$ and $(h_\epsilon(\epsilon^{-1}x), m_\epsilon(\epsilon^{-1}x))$ converges in sup-norm as $\epsilon \rightarrow 0$ to the pair $(h(x), m(x))$ solution of the Stefan problem (2.17). Moreover h_ϵ and m_ϵ are both strictly increasing if $j < 0$ and strictly decreasing if $j > 0$.*

Remarks. (a) Theorem 2.1 is proved in Section 3 and in Appendix A, B and C. The proof is based on finding the fixed point of the following map: given a function h solve the first one in (2.23) to get m and use the second one to find the new h . Existence of a fixed point is proved by showing convergence of the iterates h_n and of the corresponding m_n . Since $x_0 = 0$ if we start with an antisymmetric function, the whole orbit remains antisymmetric and indeed the limit macroscopic solution is antisymmetric as well. As we shall see restricting to the space of odd functions greatly simplifies the problem. We start the iteration from a profile m_0 which is almost a fixed point: m_0 is in fact the [scaled by ϵ^{-1}] macroscopic solution away from 0 while it is equal to the “instanton” (see Section 3) in a neighborhood of 0. We shall prove that all the profiles m_n obtained by iterating (2.23) are contained in a small neighborhood of m_0 and that the iterates converge to a limit profile m ; also the corresponding magnetic fields h_n are proved to converge to a limit h and the pair (h, m) is the desired fixed point which solves (2.23). The crucial point in the analysis is to control the change δm of m in the first equality in (2.23) when we slightly vary h by δh . To linear order δm and δh are related by $(A_{h,m} - 1)\delta m = -p_{h,m}\delta h$ where $A_{h,m} = p_{h,m}J^*$, J^* the convolution operator with kernel J , and

$$p_{h,m} = \frac{\beta}{\cosh^2\{\beta J^{\text{neum}} * m + \beta h\}} \quad (2.24)$$

$$p_{h,m} = \chi(m) \quad \text{if } m = \tanh\{\beta J^{\text{neum}} * m + \beta h\} \quad (2.25)$$

(the equality $p_{h,m} = \chi(m)$ in (2.25) will be often exploited in the sequel). Thus $\delta m = L_{h,m}^{-1}(-p_{h,m}\delta h)$ provided $L_{h,m} := A_{h,m} - 1$ is invertible. In [4] it is shown that the largest eigenvalue of $L_{h,m}$ converges to 0 as $\epsilon \rightarrow 0$ and that there is a spectral gap bounded away from 0 uniformly in ϵ . By restricting to odd functions the leading eigenvalue disappears and the invertibility problem can then be solved. As clear from this outline the proof does not give uniqueness which is left open.

(b) The choice of Neumann conditions simplifies the analysis but other conditions (provided they preserve antisymmetry) may be treated as well unless they contrast with the macroscopic value of the magnetization imposed by j , in which case boundary layers may appear, which are instead absent with Neumann conditions.

(c) With Neumann conditions the non local convolution term is completely defined, but since the evolution involves also derivatives other conditions are needed

to determine the solution: our choice was to fix j and x_0 . Dirichlet conditions would instead prescribe the limits m_{\pm} of $m(x)$ as $x \rightarrow \pm\epsilon^{-1}\ell$. There are here two types of boundary conditions, those which fix m outside the domain and are used to define the convolution (in our case replaced by Neumann conditions) and those which prescribe the values of m when going to the boundary from the interior (in our case are replaced by j and x_0). The distinction is not as clear in other models as for instance in the Cahn-Hilliard equation where more parameters are involved, we are indebted to N. Alikakos and G. Fusco for many enlightening discussions on such issues.

(d) In this paragraph it is convenient to refer to Dirichlet boundary conditions. It follows immediately from (2.14) that the critical points of the functional are stationary, i.e. such that the derivative vanishes, $DF_{\beta,\Lambda}(m) = 0$, $\Lambda = \epsilon^{-1}[-\ell, \ell]$. They are in fact special solutions of (2.23): those with $j = 0$ and hence $h = 0$, thus solutions of the mean field equation $m = \tanh\{\beta J * (m + m_{\Lambda^c})\}$, when m_{Λ^c} is fixed outside Λ . In this case the limit values of m when $x \rightarrow \partial\Lambda$ from the interior cannot be prescribed independently, they are generally different from those obtained going to $\partial\Lambda$ from the outside by using m_{Λ^c} . If we want different boundary values (from the inside) than those produced by solving $DF_{\beta,\Lambda}(m) = 0$, we must look for solutions with a current and we are back to the problem considered in this paper. The solutions of the Dirichlet problem with and without currents are qualitatively different. In the former case there is a sensitive dependence on the boundary values, even macroscopically away from the boundaries while, when the current is zero, we see the familiar exponential relaxation towards the stable phases.

We have a slightly weaker result when $x_0 \neq 0$ as we need in our proofs to replace the condition $h(\epsilon^{-1}x_0) = 0$ by an integral one, namely $\int_{-\epsilon^{-1}\ell}^{\epsilon^{-1}\ell} hu^* = 0$, where u^* (whose dependence on ϵ is not made explicit) is a suitable positive function on \mathbb{R} , symmetric around $\epsilon^{-1}x_0$ and which decays exponentially as $|x - \epsilon^{-1}x_0| \rightarrow \infty$ uniformly in ϵ (if u^* were a delta we would then be back to the condition $h(\epsilon^{-1}x_0) = 0$). We do not control the exact mesoscopic location of the zeroes of the magnetization profile m and of the magnetic field profile h , however they differ from $\epsilon^{-1}x_0$ by quantities which vanish faster than any power of ϵ as $\epsilon \rightarrow 0$:

Theorem 2.2. *Let $j \neq 0$, $\ell \in (0, \ell_j)$ and $x_0 \neq 0$ in $(-\ell, \ell)$. Then for any $\epsilon > 0$ small enough there is a pair (h_ϵ, m_ϵ) which solves (2.23) in $\epsilon^{-1}(-\ell, \ell)$. $h_\epsilon(x_\epsilon) = 0$ where $x_\epsilon \in \epsilon^{-1}(-\ell, \ell)$ and $\epsilon x_\epsilon \rightarrow x_0$ (see (G.26) in Appendix G). Finally $(h_\epsilon(\epsilon^{-1}x), m_\epsilon(\epsilon^{-1}x)) \rightarrow (h(x), m(x))$ in sup-norm as $\epsilon \rightarrow 0$, (h, m) the solution of the Stefan problem (2.17).*

Theorem 2.2 is proved in Section 4 and in Appendix D, E, F and G where we derive explicit bounds on the speed of convergence. By Theorem 2.1 we can construct a quasi solution (h_0, m_0) of (2.23) with an error which around the interface $\epsilon^{-1}x_0$ is exponentially small in ϵ^{-1} (we shall exploit this with the introduction of suitable weighted norms). (h_0, m_0) is then used as the starting point of an iterative scheme similar to the one in the proof of Theorem 2.1 from which however it differs significantly due to the absence of symmetries. The problem is that we cannot restrict

anymore to the space of antisymmetric functions and thus need to check that the maximal eigenvalue of the operator L obtained by linearizing the first equation in (2.23) is non zero. We know however from [4] that it is close to zero and actually vanishes as $\epsilon \rightarrow 0$. But in our specific case we can be more precise and prove that it is negative and bounded away from 0 proportionally to ϵ . Thus we can invert L but get a dangerous factor ϵ^{-1} in the component along the direction of the maximal eigenvector which spoils the iterative scheme as it is and it thus needs to be modified. The idea roughly speaking is to slightly shift from $\epsilon^{-1}x_0$ to make smaller the component along the maximal eigenvector (hence the condition $\int_{-\epsilon^{-1}\ell}^{\epsilon^{-1}\ell} hu^* = 0$ mentioned before Theorem 2.2) and this is enough to make the iteration work. The shifts described above are responsible for the delocalization of the zero of the magnetization profile which may not coincide with that of the magnetic field.

The Dirichlet problem.

By Theorem 2.1 and 2.2 it then follows that there are solutions of the stationary mesoscopic equation which converge as $\epsilon \rightarrow 0$ to the solution of any Dirichlet problem with $m_- < -m_\beta$ and $m_+ > m_\beta$ or viceversa. At the mesoscopic level, though, the boundary values may differ from the prescribed ones but the difference is infinitesimal in ϵ . We omit the proof that the above extends to any choice of m_\pm in the complement of $(-m_\beta, m_\beta)$ provided $m_+ \neq m_-$. We thus have a complete theory of the derivation of the Stefan problem from (2.23) gaining a deeper insight on the sense in which the values in $(-m_\beta, m_\beta)$ are forbidden. At the mesoscopic level in fact such a restriction is absent and in the approximating profiles (h_ϵ, m_ϵ) which at each ϵ solve (2.23), the values in $(-m_\beta, m_\beta)$ are indeed present in m_ϵ . However the fraction of space where they are attained becomes negligible as $\epsilon \rightarrow 0$, they concentrate at the interface which in macroscopic units becomes a point and in mesoscopic units are described to leading order by the instanton which converges exponentially fast to $\pm m_\beta$.

Under-cooling and over-heating effects.

In the forbidden interval $(-m_\beta, m_\beta)$ we distinguish two regions: one called “spinodal” is $[-m^*, m^*]$, $m^* = \sqrt{1 - 1/\beta}$, the other, $\{m_\beta > |m| > m^*\}$, is called metastable and it splits into two disjoint intervals, the plus and minus metastable phases according to the sign of m . In the spinodal region ϕ_β is concave, see (2.2), while in $(m^*, 1)$ [as well as in $(-1, -m^*)$] ϕ_β is strictly convex. If we could restrict to $(m^*, 1)$ [or to $(-1, -m^*)$] ignoring or deleting the complement, then ϕ_β would be convex and it could play the role of a thermodynamically well defined free energy giving rise to a new “metastable thermodynamics”, new because in the interval (m^*, m_β) it differs from the “true” thermodynamic free energy a_β . When (if ever) is it correct to use the metastable one? The usual answer (as its name suggests) is that the time scale should not be too long and the initial state of the system entirely in the plus [or in the minus] metastable phase. When the evolution is given by (2.19) initial states entirely in the plus phase $(m^*, 1)$ [or in the minus one, $(-1, -m^*)$] evolve remaining in the plus [minus] phase, so that the other values of

the magnetization never enter into play and can be ignored. In particular if $m_\epsilon(x, t)$ solves (2.19) with initial datum $m_\epsilon(x, 0) = m_0(\epsilon x)$, $m_0 \in C^\infty(\mathbb{R}^d; (m^*, 1))$, then

$$\lim_{\epsilon \rightarrow 0} m_\epsilon(\epsilon^{-1}x, \epsilon^{-2}t) = m(x, t) \quad (2.26)$$

solution [with initial datum m_0] of

$$\frac{\partial m}{\partial t} = \operatorname{div} \left(D^* \operatorname{grad} m \right), \quad D^* = 1 - \beta(1 - m^2) \quad (2.27)$$

with $D^* > 0$ in $(m^*, 1)$ [and in $(-1, -m^*)$ as well]. By (2.2), $D^* = \chi \phi_\beta''$ which confirms the interpretation of ϕ_β as a free energy once we compare D^* with the expression for D_β in (2.16). This is proved in [7] where the analysis extends to the spin system with Kac potentials, if the Kac scaling parameter is suitably related to ϵ so that the time scale is ϵ^{-2} . On much longer times, which scale exponentially in ϵ^{-1} , large deviations and tunnelling effects enter into play with the metastable phase becoming unstable, see [1].

All the above deals with initial states entirely in the plus [or in the minus] phase, much less is known when they coexist. A first answer is provided in this paper, see Theorems 2.1 and 2.2, where however the coexisting plus and minus phases are the thermodynamically stable ones. In such cases the whole interval $(-m_\beta, m_\beta)$ shrinks in the thermodynamic limit to a point, not distinguishing between metastable and spinodal values (thus in agreement with the macroscopic, thermodynamics of the model) Our next theorem proves that there are also stationary solutions of (2.23) where the plus and minus metastable phases coexist.

Theorem 2.3. *Let $j > 0$ then for any positive ℓ smaller than some ℓ_j , there is an antisymmetric pair $(h_\epsilon(x), m_\epsilon(x))$ which solves the stationary problem (2.23) in $\epsilon^{-1}(-\ell, \ell)$ and such that $(h_\epsilon(\epsilon^{-1}x), m_\epsilon(\epsilon^{-1}x))$ converges in sup norm as $\epsilon \rightarrow 0$ to $(h(x), m(x))$ solution of the “metastable” Stefan problem:*

$$h^*(x) = \int_0^x \frac{-j}{\chi(m)}, \quad m = \phi_\beta'^{-1}(h^*) \quad \text{in } (-\ell, \ell) \setminus \{0\} \quad (2.28)$$

h_ϵ is strictly decreasing while, to leading orders in ϵ , m_ϵ first decreases then increases (around the origin) and then again decreases. The interval where it increases has length I_ϵ and $\epsilon I_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$.

The proof of Theorem 2.3 is completely similar to the proof of Theorem 2.1 and it is therefore omitted. We did not check that the result extends to the case $x_0 \neq 0$. The coexistence of the plus and minus metastable phases is related to the presence of a current which “stabilizes” the profile. If $j = 0$ the stationary solution would be close to an instanton except for boundary layers and in the thermodynamic limit would converge to the Wulff shape which in this case is simply $m_\beta \operatorname{sign}(x)$. We conjecture that the profiles described in Theorem 2.3 are “metastable” in the sense that an additional noise at fixed current j would make the state tunnel toward the solution with same j described in Theorem 2.1.

3. PROOF OF THEOREM 2.1

In this section we shall prove Theorem 2.1 which will be a corollary of three theorems stated below and proved later in three successive appendices. For notational simplicity we suppose $j < 0$ and, as discussed in Remark (a) after Theorem 2.1, we restrict to odd functions, so that by default in this section all functions are antisymmetric. The analysis is based on an iterative scheme which is outlined in the next two paragraphs. We shall define a sequence (h_n, m_n) which for each n satisfies the equality $m_n = \tanh\{\beta J^{\text{neum}} * m_n + \beta h_n\}$ and prove that (h_n, m_n) converges as $n \rightarrow \infty$ in sup norm to a limit (h, m) which is the desired solution of (2.23).

The starting element.

We define h_0 using (2.21) with m set equal to m_0 , m_0 the odd function defined for $x > 0$ as

$$m_0(x) = \bar{m}(x)\mathbf{1}_{[0, \xi_\epsilon]}(x) + u(\epsilon[x - \xi_\epsilon])\mathbf{1}_{(\xi_\epsilon, \epsilon^{-1}\ell]}(x) \quad (3.1)$$

where: \bar{m} is the instanton (see the paragraph *Instanton: notation and properties* in Appendix A); $\xi_\epsilon = x_\epsilon + 2n_0$, $x_\epsilon : \bar{m}(x_\epsilon) = m_\beta - \epsilon$, n_0 a large integer independent of ϵ , its value will be specified in the course of the proof of Lemma A.1; as shown in Appendix A x_ϵ scales as $\log \epsilon^{-1}$. Finally, $u(r)$, $r \in [0, \ell - \epsilon\xi_\epsilon]$, is the solution of the macroscopic equation (2.17) (which in (2.17) is denoted by m). Since h_0 is obtained from m_0 by (2.21) then

$$m_0 = \tanh\{\beta J^{\text{neum}} * m_0 + \beta h_0\} \quad (3.2)$$

a property which will be satisfied by all the elements of the sequence (h_n, m_n) . Moreover, denoting by $\|\cdot\|$ the sup-norm,

$$\sup_{\epsilon} \|m_0\| \leq c_{(3.3)} < 1 \quad (3.3)$$

because $\|\bar{m}\| \leq m_\beta$ and $\|u\| < 1$ since $\ell < \ell_j$, see (2.18) and the paragraph *Axiomatic non equilibrium macroscopic theory* in Section 2.

The iterative scheme.

As discussed in Remark (a) after Theorem 2.1, the idea is to define a transformation $h \rightarrow T(h)$ [from antisymmetric into antisymmetric functions] in two steps. We first find an antisymmetric function m such that $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and then define for $x \geq 0$

$$T(h)(x) = -\epsilon j \int_0^x \chi(m(y))^{-1}, \quad m = \tanh\{\beta J^{\text{neum}} * m + \beta h\} \quad (3.4)$$

The definition of $T(h)$ thus rests on the possibility of finding an “auxiliary function” m which solves the second equality in (3.4) and it is such that $\chi(m)^{-1}$ is integrable. By construction we already know that the auxiliary function m_0 associated to h_0 exists and $\|m_0\| \leq c_{(3.3)} < 1$ uniformly in ϵ . The crucial step will then be to prove that if h is “close” to h_0 then (at least for ϵ small enough) there is a unique m “close” to m_0 so that the second equality in (3.4) is satisfied, $\|m\| < 1$ and $T(h)$ is thus well defined (we do not have general uniqueness as we are in the phase transition regime: we cannot exclude that there are other solutions not close to m_0). We shall then prove recursively that all images $h_n = T^n(h_0)$ are well defined and close to

h_0 , while the auxiliary functions m_n are close to m_0 ; moreover $(h_n, m_n) \rightarrow (h, m)$ in sup-norm as $n \rightarrow \infty$. h will then be a fixed point of T with auxiliary function m and Theorem 2.1 will be proved.

Notation.

Our basic accuracy parameter will be ϵ^a , $a \in (0, 1)$. ϵ^a defines quantitatively the a-priori closeness to h_0 (the elements h_k in the iteration will actually be much closer to h_0 , $\|h_k - h_0\| \leq c\epsilon \log \epsilon^{-1}$):

$$\|h - h_0\| \leq \epsilon^a, \quad \|f\| := \sup_{|x| \leq \epsilon^{-1}\ell} |f(x)| \quad (3.5)$$

being understood that all functions we deal with in this section are odd. While the basic accuracy parameter clearly depends on ϵ , $a \in (0, 1)$ above as well as all the constants that we shall write in the sequel, denoted by a , b , c and C with or without suffixes, will be independent of ϵ . The existence of the auxiliary function m in (3.4) is established next:

Theorem 3.1. *There are constants $c_{(3.6)} > 1$, $\alpha_{(3.6)} > 0$, $c'_{(3.6)} := \frac{2c_{(3.6)}}{\alpha_{(3.6)}}$ so that for all ϵ small enough the following holds. For any $h : \|h - h_0\| \leq \epsilon^a$ there is a unique m_h in the ball $\{m : \|m - m_0\| \leq c'_{(3.6)}\epsilon^a\}$ such that $m_h = \tanh\{\beta J^{\text{neum}} * m_h + \beta h\}$ and for any $h' : \|h' - h_0\| \leq \epsilon^a$*

$$|m_h(x) - m_{h'}(x)| \leq c_{(3.6)} \int_0^{\epsilon^{-1}\ell} e^{-\alpha_{(3.6)}|x-y|} |h(y) - h'(y)|, \quad x \geq 0 \quad (3.6)$$

We postpone to Appendix A the proof of Theorem 3.1 and proceed with the proof of Theorem 2.1 observing that as a consequence of Theorem 3.1 if $\|h - h_0\| \leq \epsilon^a$ then $T(h)$ is well defined (for all ϵ small enough) because $\chi(m)$ in the first of (3.4) is bounded away from 0. To prove this it suffices to show that $\|m\| < 1$. By (3.6) with $m_h = m$ and $m_{h'} = m_0$,

$$\|m - m_0\| \leq c'_{(3.6)} \|h - h_0\| \leq c'_{(3.6)} \epsilon^a \quad (3.7)$$

Then by (3.3) $\|m\| \leq c_{(3.3)} + c'_{(3.6)} \epsilon^a < 1$ for ϵ small enough.

Theorem 3.2. *There are constants $c_{(3.8)}$ and $c_{(3.9)} > 0$ so that for all ϵ small enough the following holds. Let m' and m'' be both in the ball $\{m : \|m - m_0\| \leq c'_{(3.6)}\epsilon^a\}$, then denoting by $h' = \int_0^x \frac{-\epsilon j}{\chi(m')} , h'' = \int_0^x \frac{-\epsilon j}{\chi(m'')} ,$*

$$|h'(x) - h''(x)| \leq c_{(3.8)} \epsilon |j| \int_0^x |m'(y) - m''(y)|, \quad x > 0 \quad (3.8)$$

$$\|h_1 - h_0\| \leq c_{(3.9)} \epsilon \log \epsilon^{-1}, \quad h_1 = T(h_0) \quad (3.9)$$

We postpone to Appendix B the proof of Theorem 3.2 and observe that since the transformation T is well defined in the ball $\|h - h_0\| \leq \epsilon^a$ we are in business once we

show that any iterate of T is in the ball $\|h - h_0\| \leq \epsilon^a$. We postpone to Appendix C the proof of:

Theorem 3.3. *There is a constant $c_{(3.10)} > 0$ so that the following holds. Suppose there is n such that for all $k < n$, $h_k = T^k(h_0)$ is well defined, $\|h_k - h_0\| \leq \epsilon^a$ and $\|m_k - m_0\| \leq \epsilon^a$, m_k the auxiliary function in the definition of $T(h_k)$. Then h_n is well defined and*

$$\|h_{k+1} - h_k\| \leq c_{(3.10)} \left(\frac{1}{2}\right)^k \|h_1 - h_0\|, \quad k < n \quad (3.10)$$

It is now easy to prove Theorem 2.1. We restrict to $\epsilon > 0$ so small that

$$2c_{(3.10)}c_{(3.13)}c_{(3.9)}\epsilon \log \epsilon^{-1} < \epsilon^a \quad (3.11)$$

(with $c_{(3.13)}$ defined in (3.13) below) and prove by induction that (h_k, m_k) exists for all k and moreover $\|h_k - h_0\| \leq \epsilon^a$ and $\|m_k - m_0\| \leq \epsilon^a$. Since the statement is obviously true for $k = 0$ we only need to prove that if it is verified for $k < n$ then it holds for n as well. By (3.10) and (3.9) for all $k < n$,

$$\|h_{k+1} - h_k\| \leq c_{(3.10)} \left(\frac{1}{2}\right)^k c_{(3.9)} \epsilon \log \epsilon^{-1} \quad (3.12)$$

which, by (3.11) shows that $\|h_n - h_0\| < \epsilon^a$ (for ϵ small enough). Then by Theorem 3.1 m_n is well defined and by (3.6) for all $k < n$

$$\|m_{k+1} - m_k\| \leq c_{(3.13)} \|h_{k+1} - h_k\|, \quad c_{(3.13)} = \max\{1, c'_{(3.6)}\} \quad (3.13)$$

Then, using (3.12),

$$\|m_{k+1} - m_k\| \leq c_{(3.10)}c_{(3.13)} \left(\frac{1}{2}\right)^k c_{(3.9)} \epsilon \log \epsilon^{-1} \quad (3.14)$$

which by (3.11) proves that $\|m_n - m_0\| \leq \epsilon^a$. Thus the induction is proved and we know that for all k , (h_k, m_k) exists, $\|h_k - h_0\| \leq \epsilon^a$ and $\|m_k - m_0\| \leq \epsilon^a$.

As a consequence of (3.14) and (3.12), $(h_n, m_n) \rightarrow (h, m)$ in sup-norm with $h = T(h)$, $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$, and

$$\|h - h_0\| \leq c\epsilon \log \epsilon^{-1}, \quad \|m - m_0\| \leq c\epsilon \log \epsilon^{-1} \quad (3.15)$$

Making explicit the dependence on ϵ we write the limit as (h_ϵ, m_ϵ) in agreement with the notation in Theorem 2.1. Recalling the definition of (h_0, m_0) , see (3.1), we then obtain the proof of Theorem 2.1 except for the statement about the monotonicity of m_ϵ which is proved at the end of Appendix D.

4. OUTLINE OF THE PROOF OF THEOREM 2.2

The macroscopic solution.

For the sake of definiteness we suppose $j < 0$ and $x_0 > 0$ and for notational simplicity that the interval $(-\ell, \ell)$ is just the interval $(-1, 1)$. By assumption $(-1, 1)$ is then strictly contained in the interval of length $2\ell_j$ and center x_0 , i.e. the maximal interval where the macroscopic problem with parameters (j, x_0) has solution (see

the paragraph *Axiomatic non equilibrium macroscopic theory* in Section 2). We then write $\ell^* = 1 + 2x_0$ so that x_0 is the middle point of the interval $[-1, \ell^*]$ and, for what said above, $\ell^* + 1 < 2\ell_j$ so that the macroscopic problem has a solution $(h_{\text{mac}}(x), m_{\text{mac}}(x))$, $x \in (-1, \ell^*)$ with the following properties: it is a smooth pair of functions antisymmetric around x_0 such that $\|m_{\text{mac}}\| < 1$ and $\|h_{\text{mac}}\| < \infty$ (so that $\inf \chi(m_{\text{mac}}) > 0$).

The pairs (h^, m^*) and (h_ϵ, m_ϵ) .*

By Theorem 2.1 for any $\epsilon > 0$ small enough there is a pair $(h^*(x), m^*(x))$, $x \in \epsilon^{-1}[-1, \ell^*]$ (dependence on ϵ is not made explicit) which solves (2.23) and is antisymmetric around $\epsilon^{-1}x_0$. Then there is $c_{(4.1)} > 0$ so that

$$\beta \geq p_{h^*, m^*} \geq c_{(4.1)} \quad \text{for all } \epsilon > 0 \text{ small enough} \quad (4.1)$$

$\beta \geq p_{h, m}$ is true in general, see (2.24); instead $p_{h^*, m^*} \geq c_{(4.1)}$ because by (2.25) $p_{h^*, m^*} = \chi(m^*) = \beta(1 - (m^*)^2)$ and $\|m^*\| < 1$ uniformly in ϵ . This follows from the inequality $\|m_{\text{mac}}\| < 1$ because, by Theorem 2.1, $\lim_{\epsilon \rightarrow 0} \|m_{\text{mac}}(x) - m^*(\epsilon^{-1}x)\| = 0$. We next define (h_ϵ, m_ϵ) :

$$\begin{aligned} m_\epsilon(x) &= m^*(x), \quad h_\epsilon(x) = h^*(x) + R_\epsilon(x), \quad x \in \epsilon^{-1}[-1, 1] \\ R_\epsilon(x) &= \int_{\epsilon^{-1}}^{\epsilon^{-1}+1} J(x, y)[m^*(y) - m^*(2\epsilon^{-1} - y)] dy \end{aligned} \quad (4.2)$$

We have added the “correction” R_ϵ to have:

$$m_\epsilon = \tanh\{\beta[J^{\text{neum}} * m_\epsilon] + \beta h_\epsilon\} \quad (4.3)$$

Lemma 4.1. *There are $r_{(4.4)} > 0$, $c_{(4.4)} > 0$ and $c'_{(4.4)} > 0$ so that*

$$\left\| \frac{dm^*}{dx} \right\| \leq c'_{(4.4)}, \quad \sup_{|x - \epsilon^{-1}x_0| \geq r_{(4.4)} \log \epsilon^{-1}} \left| \frac{dm^*(x)}{dx} \right| \leq c_{(4.4)}\epsilon \quad (4.4)$$

As a consequence $|R_\epsilon(x)| \leq c\epsilon \mathbf{1}_{\epsilon^{-1}-1 \leq x \leq \epsilon^{-1}}$.

Proof. By differentiating the equality $m^* = \tanh\{\beta J^{\text{neum},*} * m^* + \beta h^*\}$ (valid in the whole interval $\epsilon^{-1}(-1, \ell^*)$, $J^{\text{neum},*}$ the kernel with Neumann conditions at its endpoints) we get

$$\left\| \frac{dm^*}{dx} \right\| \leq \beta \left(\left\| \frac{dJ^{\text{neum},*}}{dx} \right\| \|m^*\| + \left\| \frac{dh^*}{dx} \right\| \right) \leq c$$

because $\|dh^*/dx\| \leq c\epsilon$ (as h^* solves (2.23)), hence the first inequality in (4.4). The second one is not as easy and it will be proved at the end of Appendix D. Using such inequality in (4.2) we readily see that $|R_\epsilon(x)| \leq c\epsilon \mathbf{1}_{\epsilon^{-1}-1 \leq x \leq \epsilon^{-1}}$, $c = c_{(4.4)}$. \square

Thus R_ϵ is “a small boundary field” and except for the small error R_ϵ , $\chi(m_\epsilon) \frac{dh_\epsilon}{dx} = -\epsilon j$ so that the pair (h_ϵ, m_ϵ) is “almost a solution” of the stationary problem (which could be interpreted as a true solution of a problem with suitably redefined boundary conditions).

An interpolation scheme.

A natural way to obtain a true solution from a quasi solution is via the implicit function theorem after writing (2.23) as a single equation $f(h, m) = 0$ on the space of pairs of L^∞ functions. Unfortunately we do not have a good control of the derivative of $f(h, m)$ which may in principle vanish. The problem simplifies if we try to solve only the first one in (2.23) and then use the second one to redefine h , which opens the way to an iterative scheme as the one used in Section 3. The crucial step is the following: find \tilde{m} such that $\tilde{m} = \tanh\{\beta J^{\text{neum}} * \tilde{m} + \beta \tilde{h}\}$ knowing \tilde{h} and that \tilde{h} is “close” to another field \hat{h} for which there is \hat{m} such that $\hat{m} = \tanh\{\beta J^{\text{neum}} * \hat{m} + \beta \hat{h}\}$. To solve this problem we interpolate writing $h(t) = t\tilde{h} + (1-t)\hat{h}$, $t \in [0, 1]$, and pretending that for all t there is $m(t)$ such that $m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$, we differentiate and get an equation for dm/dt . Its solution will then allow to obtain \tilde{m} as $\tilde{m} = \hat{m} + \int_0^1 \frac{dm}{ds} ds$.

The main point in this procedure is therefore the analysis of the equation for dm/dt . This is (E.2) in Appendix E, here we just say that it has the form $\psi = (A_{h,m} - 1)^{-1} \phi$ (ψ the unknown), where $A_{h,m} = p_{h,m} J^{\text{neum}} *$, $p_{h,m}$ as in (2.24) (and $p_{h,m} = \chi(m) = \beta(1-m^2)$ because $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$), $J^{\text{neum}} *$ is the convolution operator with kernel J^{neum} . The non linearity of the problem reflects in the fact that (h, m) above is actually $(h(t), m(t))$ which is itself unknown but the whole problem boils down to an accurate analysis of the operator $A_{h,m}$ in a suitably large set of pairs (h, m) (the set \mathcal{A} in Appendix D). The same problem has appeared in the proof of Theorem 3.1, where however we had the great simplification to restrict to the space of antisymmetric functions. In such a restricted space $\|A_{h,m}^{n_0}\| < 1$ for a suitable integer n_0 uniformly in ϵ (see Appendix A). $(1 - A_{h,m})^{-1}$ is then equal to the convergent sum $\sum A_{h,m}^n$ and the bound (A.13) holds. In the case considered in Theorem 2.2 we do not have symmetries and the invertibility of $L_{h,m} := A_{h,m} - 1$ becomes a serious issue.

In Appendix D we shall establish fine spectral properties of $A_{h,m}$ for all (h, m) in a set \mathcal{A} . We shall prove a Perron-Frobenius theorem for $A_{h,m}$ regarded as an integral operator on $L^\infty(\epsilon^{-1}[-1, 1])$ showing that it has a maximal eigenvalue $\lambda > 0$, that its eigenvector u (called the “maximal eigenvector”) has a definite sign (taken positive) and, see Proposition D.1, that there are positive constants c , c' and a so that for all ϵ small enough

$$0 < \lambda < 1 - c\epsilon, \quad 0 < u \leq c' e^{-a|x-x_0|} \quad (4.5)$$

Actually to leading order in ϵ , $\lambda = 1 - C_{(\text{D.9})}\epsilon$, see (D.9). λ is separated from the rest of the spectrum (spectral gap) as stated in Proposition D.2.

Our strategy therefore will be to reduce to pairs $(h, m) \in \mathcal{A}$, a task accomplished by showing that we can actually reduce to functions h in the very small neighborhood \mathcal{G} of h_ϵ defined next.

The set \mathcal{G}

Let $b_{(4.7)}$ and $a_{(4.6)}$ be positive parameters (specified in Appendix F), and for any $f \in L^\infty(\epsilon^{-1}[-1, 1])$

$$N(f) := \sup_{|x| \leq \epsilon^{-1}} E_\epsilon(x) |f(x)|; \quad E_\epsilon(x) := \begin{cases} e^{a_{(4.6)}(\epsilon^{-1}-x)} & x \geq \epsilon^{-1}x_0 \\ e^{a_{(4.6)}^-(x+\epsilon^{-1})} & x < \epsilon^{-1}x_0 \end{cases} \quad (4.6)$$

with $a_{(4.6)}^-$ such that $a_{(4.6)}^-(x_0 + 1) = a_{(4.6)}(1 - x_0)$. Recalling that h_ϵ is defined in (4.2) and denoting by $u^* \in L^\infty(\epsilon^{-1}[-1, \ell^*], \mathbb{R}^+)$ the “maximal eigenvector” of A_{h^*, m^*} we define \mathcal{G} as

$$\mathcal{G} := \left\{ h : N(h - h_\epsilon) \leq b_{(4.7)}, \int_{-\epsilon^{-1}}^{\epsilon^{-1}} h u^* = 0, \left\| \frac{d(h - h_\epsilon)}{dx} \right\| \leq \epsilon \sup_{|x - \epsilon^{-1}x_0| \leq (\log \epsilon^{-1})^2} \left| \frac{d(h - h_\epsilon)}{dx} \right| \leq \epsilon^2 \right\} \quad (4.7)$$

The iterative scheme.

We shall prove in Proposition F.1 that if $h \in \mathcal{G}$ then there is m such that $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and moreover $(h, m) \in \mathcal{A}$, \mathcal{A} the nice set with good spectral properties mentioned earlier. Thus $A_{h, m}$ has a maximal eigenvalue λ with maximal eigenvector u , $A_{h, m}u = \lambda u$. In Corollary F.4 we shall prove that u is “very close” to the restriction of u^* to $\epsilon^{-1}[-1, 1]$, u^* the maximal eigenvector of A_{h^*, m^*} relative to the problem in $\epsilon^{-1}[-1, \ell^*]$. All this collects the properties needed to define the iterative scheme and to prove its convergence. We define recursively $h_{n+1} := T(h_n)$, $n \geq -1$, $h_{-1} := h_\epsilon$, as

$$h_{n+1} = \hat{h}_{n+1} - \frac{\int_{-\epsilon^{-1}}^{\epsilon^{-1}} \hat{h}_{n+1} u^*}{\int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^*}, \quad \hat{h}_{n+1}(x) := -\epsilon j \int_{\epsilon^{-1}x_0}^x \chi(m_n(y))^{-1} \quad (4.8)$$

(recalling that $\chi(m_n) = p_{h_n, m_n}$ by (2.25)). The definition is well posed once we prove that $h_n \in \mathcal{G}$ for $n \geq 0$, so that there is a unique m_n such that $(h_n, m_n) \in \mathcal{A}$. We shall indeed prove in Proposition G.3 that $N(h_{n+1} - h_n) \leq c\epsilon N(h_n - h_{n-1})$. Here we use in an essential way the subtraction in (4.8) which subtracts [most of] the component along the maximal eigenvector u of $A_{h_{n-1}, m_{n-1}}$ of the “forcing term” $p_{h_{n-1}, m_{n-1}}(h_n - h_{n-1})$. In this way we shall prove iteratively that $h_n \in \mathcal{G}$ so that there is m_n with $(h_n, m_n) \in \mathcal{A}$; moreover we shall see in Appendix G that $h_n \rightarrow h$ and $m_n \rightarrow m$ as $n \rightarrow \infty$ with $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$, $h = \hat{h} - \frac{\int \hat{h} u^*}{\int u^*}$, $\hat{h}(x) := -\epsilon j \int_{\epsilon^{-1}x_0}^x \chi(m(y))^{-1}$. As a consequence the pair (h, m) satisfies (2.22) with $h(x_\epsilon) = 0$ where x_ϵ is such that:

$$\int_{\epsilon^{-1}x_0}^{x_\epsilon} \chi(m(y))^{-1} = \frac{\int u^*(x) \int_{\epsilon^{-1}x_0}^x \chi(m(y))^{-1}}{\int u^*} \quad (4.9)$$

The proof of Theorem 2.2 will then be completed by showing at the end of Appendix G that x_ϵ exists and that $\epsilon x_\epsilon \rightarrow x_0$ as $\epsilon \rightarrow 0$, see (G.26).

APPENDIX A. PROOF OF THEOREM 3.1

Before proving Theorem 3.1 we introduce some notation and definitions which will be used throughout the whole sequel.

An auxiliary dynamics

To construct and compare solutions of $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ for given h , we introduce an artificial dynamics. Suppose $(h(t), m(t))$, $t \in [0, 1]$, are smooth functions of t and that for all t

$$m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\} \quad (\text{A.1})$$

By differentiating (A.1) with respect to t we get the identity

$$\frac{dm}{dt} = A_t \frac{dm}{dt} + p_t \frac{dh}{dt}, \quad L_t \frac{dm}{dt} = -p_t \frac{dh}{dt}, \quad L_t = A_t - 1 \quad (\text{A.2})$$

where $p_t = p_{h(t), m(t)}$, $p_{h, m}$ as in (2.24), and $A_t = p_t J^{\text{neum}} *$, $J^{\text{neum}} *$ the operator on $L^\infty(\epsilon^{-1}[-\ell, \ell])$ with kernel J^{neum} .

By a change of perspective we now regard (A.2) as an equation for the unknown $\frac{dm}{dt}$ with p_t and $\frac{dh}{dt}$ considered as “known terms”. We shall prove in this appendix that under suitable assumptions on h a solution exists and it is unique. We then “construct” $m(t) := m(0) + \int_0^t \frac{dm}{ds}$ and check that it verifies (A.1). The important point is that the whole procedure works in the same way even if we ask that (A.1) holds only at time $t = 0$, being a by-product of the analysis that it remains valid for all $t \in [0, 1]$. In the actual applications $m(0) = m_0$ is a given, known function which solves $m_0 = \tanh\{\beta J^{\text{neum}} * m_0 + \beta h_0\}$, $h(t) = h_1 t + (1 - t)h_0$ with h_0 and h_1 also known and $m(t)$ the unknown, in particular we are interested in its value m_1 at time $t = 1$ when $h(1) = h_1$. (A.2) then becomes a non linear evolution equation and it will be crucial to prove first that L_t is invertible, so that the equation can be written in normal form

$$\frac{dm}{dt} = L_t^{-1} \left(-p_t \frac{dh}{dt} \right) \quad (\text{A.3})$$

and then that $L_t^{-1} \left(-p_t \frac{dh}{dt} \right)$ is a Lipschitz function of m .

The operator $A_{h, m}$

The whole analysis relies on properties of the spectrum of the operator $A_{h, m} = p_{h, m} J^{\text{neum}} *$ (called A_t when $(h, m) = (h(t), m(t))$ as above). We shall study $A_{h, m}$ in a $L^\infty(\epsilon^{-1}[-\ell, \ell])$ setting and since we want to prove that $A_{h, m} - 1$ is invertible it is crucial to prove that 1 is not in the spectrum of $A_{h, m}$. Regarded as an operator on $L^2(\epsilon^{-1}[-\ell, \ell], p_{h, m}^{-1}(x)dx)$, $A_{h, m}$ is self-adjoint, it has a maximal eigenvalue $\lambda_{h, m}$ which is positive and the corresponding eigenvector $u_{h, m}$, called the maximal eigenvector, can and will be chosen as strictly positive, see [4]. Further assumptions on h and m will allow to prove that $\lambda_{h, m} \leq 1 - c\epsilon$, $c > 0$, and that the rest of the spectrum is strictly below 1 uniformly in ϵ . The bound on $\lambda_{h, m}$ will not be used in this appendix, see the proof of (A.8) below.

Instanton: notation and properties

The instanton \bar{m} is a solution of the local mean field equation $\bar{m}(x) = \tanh\{\beta J * \bar{m}(x)\}$, $x \in \mathbb{R}$, with the following properties (see Section 8.1 and 8.2 of [9]). $\bar{m}(x)$ is a strictly increasing, antisymmetric function which converges to $\pm m_\beta$ as $x \rightarrow \pm\infty$, more precisely there are $c_{(A.4)}$ and $a_{(A.4)}$ both positive so that for all $x \geq 0$

$$0 < m_\beta - \bar{m}(x) \leq c_{(A.4)} e^{-a_{(A.4)}x}, \quad \frac{d\bar{m}(x)}{dx} \leq c_{(A.4)} e^{-a_{(A.4)}x} \quad (A.4)$$

We write

$$\bar{p} = \beta(1 - \bar{m}^2), \quad \bar{A} = \bar{p}J*, \quad \bar{m}' = \frac{d\bar{m}}{dx}, \quad \tilde{m}' = \frac{\bar{m}'}{\langle (\bar{m}')^2 \rangle_\infty^{1/2}} \quad (A.5)$$

where $\langle f \rangle_\infty = \int_{\mathbb{R}} f \bar{p}^{-1}$. In [4] and Section 8.3 in [9] it is proved that there are $a_{(A.6)} > 0$ and $c_{(A.6)}$ so that for any bounded function f

$$\left| \int \bar{A}^n(x, y) \tilde{f}(y) dy \right| \leq \| \tilde{f} \| c_{(A.6)} e^{-a_{(A.6)}n}, \quad \tilde{f} = f - \langle f \tilde{m}' \rangle_\infty \tilde{m}' \quad (A.6)$$

We can now turn to the proof of Theorem 3.1 and restrict hereafter in this appendix to the space of antisymmetric functions. After observing that by (A.4)

$$x_\epsilon \leq c \log \epsilon^{-1} \quad (A.7)$$

we complete the definition (3.1) of m_0 by fixing the integer n_0 , chosen so that

$$\| \bar{A}^{n_0} \psi \| \leq e^{-a_{(A.8)}} \| \psi \|, \quad a_{(A.8)} > 0 \quad (A.8)$$

where ψ above is any bounded antisymmetric function, recall that $\|f\|$ denotes the sup norm of f . Existence of n_0 follows from (A.6) because \bar{m}' and \bar{p}_{x_0} are symmetric and ψ antisymmetric so that $\langle \psi \bar{m}' \rangle_\infty = 0$.

Lemma A.1. *There is $a_{(A.9)} > 0$ so that for any $c, a > 0$ and all ϵ small enough*

$$\| A_{h,m}^{n_0} \psi \| \leq e^{-a_{(A.9)}} \| \psi \|, \quad \text{if } \| m - m_0 \| \leq c\epsilon^a, \quad \| h - h_0 \| \leq c\epsilon^a \quad (A.9)$$

for any bounded odd function ψ .

Proof. As we shall see (A.9) is a straight consequence of (A.8) and of

$$\left\| \frac{p_{h,m}}{p_{h_0,m_0}} - 1 \right\| \leq c' \epsilon^a \quad (A.10)$$

which follows directly from (3.3) and the assumptions on h and m . We distinguish “small” and “large” values of x_0 in $A_{h,m}^{n_0} \psi(x_0)$.

(i). $x_0 \in [0, x_\epsilon + n_0]$. We write

$$A_{h,m}^{n_0} \psi(x_0) = \int \psi(x_{n_0}) \prod_{k=1}^{n_0} \{ p_{h_0,m_0}(x_{k-1}) J^{\text{neum}}(x_{k-1}, x_k) \frac{p_{h,m}(x_{k-1})}{p_{h_0,m_0}(x_{k-1})} \} dx_1 \cdots dx_{n_0} \quad (A.11)$$

Since J^{neum} has range 1, $|x_i| \leq x_\epsilon + 2n_0$ for all $i = 1, \dots, n_0$. Then by (A.7) for ϵ small enough, $J^{\text{neum}}(x_i, x_{i+1}) = J(x_i, x_{i+1})$. Moreover $p_{h_0, m_0}(x_i) = \bar{p}(x_i)$ (because $m_0(x) = \bar{m}(x)$, $h_0(x) = 0$ for $|x| \leq x_\epsilon + 2n_0$). Thus by (A.10)

$$\left| A_{h,m}^{n_0} \psi(x_0) - \bar{A}^{n_0} \psi(x_0) \right| \leq c' n_0 \epsilon^a \|\psi\|$$

and using (A.8), for all ϵ small enough

$$\left| A_{h,m}^{n_0} \psi(x_0) \right| \leq e^{-a(\text{A.8})} \|\psi\| + c' n_0 \epsilon^a \|\psi\| \leq e^{-a(\text{A.9})} \|\psi\|$$

(ii). $x_0 \in [x_\epsilon + n_0, \epsilon^{-1}\ell]$. We then write

$$A_{h,m}^{n_0} \psi(x_0) = \int \psi(x_{n_0}) \prod_{k=1}^{n_0} \{p_{h,m}(x_{k-1}) J^{\text{neum}}(x_{k-1}, x_k) dx_1 \cdots dx_{n_0}\} \quad (\text{A.12})$$

and since J^{neum} has range 1, $x_i \geq x_\epsilon$ in (A.12) for all $i = 1, \dots, n_0$. When $x_i \in [x_\epsilon, \xi_\epsilon]$, $p_{h_0, m_0}(x_i) = \bar{p}(x_i) \leq \beta(1 - \bar{m}(x_\epsilon)^2)$ and by the definition of x_ϵ , $\bar{m}(x_\epsilon) = m_\beta - \epsilon$. Hence if b' is such that $\beta(1 - m_\beta^2) < b' < 1$ then for all ϵ small enough, $p_{h_0, m_0}(x_i) \leq b' < 1$. When $x_i > \xi_\epsilon$, $p_{h_0, m_0} = \beta(1 - u^2)$ and since $u \geq m_\beta$, $p_{h_0, m_0}(x_i) \leq \beta(1 - m_\beta^2) < b' < 1$. Thus by (A.10) $p_{h,m}(x_i) \leq b < 1$ $|A_{h,m}^{n_0} \psi(x_0)| \leq b^{n_0} \|\psi\|$. \square

By (A.9), $L_{h,m} = A_{h,m} - 1$ is invertible and

$$L_{h,m}^{-1} = - \sum_{n=0}^{\infty} A_{h,m}^n, \quad \|L_{h,m}^{-1}\| \leq \frac{c(\text{A.13})}{1 - a(\text{A.9})} \quad (\text{A.13})$$

where $c(\text{A.13})$ bounds $\sum_{n=1}^{n_0} \|A_{h,m}^n\|$. Moreover:

Lemma A.2. *There exist $\alpha_{(3.6)} > 0$, (which defines the parameter introduced in (3.6)), $c_{(\text{A.14})}$ and $c_{(\text{A.15})}$, both larger than $\max\{1, \frac{c(\text{A.13})}{1 - a(\text{A.9})}\}$, so that for any c and all ϵ small enough*

$$|L_{h,m}^{-1} \psi(x)| \leq c_{(\text{A.14})} \int_0^{\epsilon^{-1}} e^{-\alpha_{(3.6)} |x-y|} |\psi(y)|, \quad \|m - m_0\| \leq c\epsilon^a, \quad \|h - h_0\| \leq c\epsilon^a \quad (\text{A.14})$$

for any $x \geq 0$. Moreover if also $m' : \|m' - m_0\| \leq c\epsilon^a$, then

$$\|L_{h,m}^{-1} - L_{h,m'}^{-1}\| \leq c_{(\text{A.15})} \|m - m'\| \quad (\text{A.15})$$

Proof. To prove (A.14) we write $A_{h,m}^n(x, y)$ as the kernel of $A_{h,m}^n$ and have by (A.13)

$$L_{h,m}^{-1} \psi(x) = - \sum_{n=0}^{\infty} \int A_{h,m}^n(x, y) \psi(y) = - \int \sum_{n=n(x,y)}^{\infty} A_{h,m}^n(x, y) \psi(y)$$

where $n(x, y) \geq |y - x|$ because $A_{h,m}(x, y) = p_{h,m}(x)J^{\text{neum}}(x, y)$ is supported by $|x - y| \leq 1$. (A.14) then follows from (A.9). To prove (A.15) we write

$$L_{h,m}^{-1} - L_{h,m'}^{-1} = L_{h,m}^{-1}(A_{h,m} - A_{h,m'})L_{h,m'}^{-1}$$

use (A.13) and that $\|A_{h,m} - A_{h,m'}\| \leq c\|m - m'\|$. \square

We shall study (A.3) with

$$h(t) = th'' + (1-t)h', \quad h' \text{ and } h'' \text{ in the ball } \|h - h_0\| \leq \epsilon^a \quad (\text{A.16})$$

so that $\|h(t) - h_0\| \leq \epsilon^a$ and $\|\frac{dh(t)}{dt}\| \leq 2\epsilon^a$. The initial datum m' is chosen so that $m' = \tanh\{\beta J^{\text{neum}} * m' + \beta h'\}$ and $\|m' - m_0\| \leq c'\epsilon^a$ where $c' := \beta c_{(\text{A.14})}$. To prove existence of solutions of (A.3) we need to control the “velocity field”

$$V(h, m, \dot{h}) = -L_{h,m}^{-1}(p_{h,m}\dot{h}) \quad (\text{A.17})$$

where m, h, \dot{h} are antisymmetric functions. To this end we specify the “free parameter” c which appears in the previous two lemmas so that $c > 3c'$, $c' := \beta c_{(\text{A.14})}$.

Lemma A.3. *For all ϵ small enough the Cauchy problem in the interval $t \in [0, 1]$*

$$\frac{dm(t)}{dt} = V(h(t), m(t), \frac{dh(t)}{dt}), \quad m(0) = m' \quad (\text{A.18})$$

*has a unique solution $m(t)$ such that $\|m(t) - m_0\| \leq 3c'\epsilon^a$. Moreover, $m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$ for all $t \in [0, 1]$.*

Proof. When $\|m - m_0\| \leq 3c'\epsilon^a$, $\|h - h_0\| \leq \epsilon^a$ the velocity field $V(h, m, \dot{h})$ is bounded (by (A.13)) and Lipschitz (by (A.15)), recall that by (A.16) $\|\dot{h}\| \leq 2\epsilon^a$. We thus have local existence and uniqueness till when $\|m - m_0\| \leq 3c'\epsilon^a$. Till this time

$$\|\frac{dm(t)}{dt}\| \leq \beta \|L_t^{-1} \frac{dh(t)}{dt}\| \leq \beta \frac{c_{(\text{A.13})}}{1 - a_{(\text{A.9})}} 2\epsilon^a \leq 2\beta c_{(\text{A.14})} \epsilon^a$$

(recalling from Lemma A.2 that $c_{(\text{A.14})} \geq \max\{1, \frac{c_{(\text{A.13})}}{1 - a_{(\text{A.9})}}\}$). Hence $\|m(t) - m(0)\| \leq 2c'\epsilon^a$ which ensures existence till $t = 1$. Recalling (A.17) we get from (A.18) that

$$\frac{d}{dt} \left(m(t) - \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\} \right) = 0$$

$m(t) - \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$ is thus constant and being 0 initially it is 0 at all times. \square

By taking $h' = h_0$ in (A.16) by Lemma A.3 we conclude that for any $h : \|h - h_0\| \leq \epsilon^a$ there is m which satisfies $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and $\|m - m_0\| \leq 3c'\epsilon^a$, $c' = \beta c_{(\text{A.14})}$.

Finally to prove (3.6) we write $h(t) = th + (1-t)h'$ so that

$$m_h - m_{h'} = \int_0^1 \frac{dm(t)}{dt} = - \int_0^1 L_t^{-1} p_t(h - h')$$

and (3.6) follows from (A.14). The proof of Theorem 3.1 is complete.

APPENDIX B. PROOF OF THEOREM 3.2

By (3.3) there is $b < 1$ so that for all ϵ small enough $\|m\| \leq b < 1$ in the ball $\{m : \|m - m_0\| \leq c_{(3.6)}\epsilon^a\}$; (3.8) then readily follows. To prove (3.9) we observe that $h_0(x) = 0$ for $x \in [0, x_\epsilon + 2n_0]$ because in such interval $m_0 = \bar{m}$ and $\bar{m} = \tanh\{\beta J^{\text{neum}} \star \bar{m}\}$. Thus, if $h_1 = T(h_0)$ by (A.7)

$$|h_1(x) - h_0(x)| \leq c\epsilon \log \epsilon^{-1}, \quad |x| \leq \xi_\epsilon := x_\epsilon + 2n_0$$

Define for $x > \xi_\epsilon$

$$h(x) = \int_{\xi_\epsilon}^x \frac{-\epsilon j}{\chi(u)} = \int_{\xi_\epsilon}^x \frac{-\epsilon j}{\chi(m_0)} \quad (\text{B.1})$$

then $m_0(x) = u(\epsilon[x - \xi_\epsilon]) = \tanh\{\beta u(\epsilon[x - \xi_\epsilon]) + \beta h(\epsilon[x - \xi_\epsilon])\}$, hence

$$m_0(x) = \tanh\{\beta J^{\text{neum}} \star m_0(x) + \beta(u(\epsilon[x - \xi_\epsilon]) - J^{\text{neum}} \star m_0(x) + h(\epsilon[x - \xi_\epsilon]))\} \quad (\text{B.2})$$

Since $m_0(x) = \tanh\{\beta J^{\text{neum}} \star m_0(x) + \beta h_0(x)\}$, by (B.2)

$$|h_0(x) - h(\epsilon[x - \xi_\epsilon])| \leq c\epsilon, \quad \text{and by (B.1)} \quad |h_0(x) - \int_{\xi_\epsilon}^x \frac{-\epsilon j}{\chi(m_0)}| \leq c\epsilon,$$

Since $h_1(x) = \int_0^x \frac{-\epsilon j}{\chi(m_0)}$, $|h_1(x) - h_0(x)| \leq |h_0(x) - \int_{\xi_\epsilon}^x \frac{-\epsilon j}{\chi(m_0)}| + c\epsilon\xi_\epsilon$ hence (3.9).

APPENDIX C. PROOF OF THEOREM 3.3

By assumption for $x \geq 0$, $0 \leq m_k(x) \leq m_0(x) + \epsilon^a$, $k < n$. By (3.3) $\|m_0\| < 1$ so that for all ϵ small enough, p_{h_k, m_k} is uniformly bounded away from 0. There is therefore $C < \infty$ (recall the current j is a constant) such that

$$|h_{k+1}(x) - h_k(x)| \leq C\epsilon \int_0^x |m_k(y) - m_{k-1}(y)| \quad (\text{C.1})$$

By (3.6) for any $y \in [0, \epsilon^{-1}]$,

$$|m_k(y) - m_{k-1}(y)| \leq c \int_0^{\epsilon^{-1}\ell} e^{-\alpha|y-z|} |h_k(z) - h_{k-1}(z)| \quad (\text{C.2})$$

where we have dropped the suffixes from c and α . We define $\psi_{k+1}(x) = |h_{k+1}(\epsilon^{-1}x) - h_k(\epsilon^{-1}x)|$, $x \in [0, \ell]$ and by combining (C.1) and (C.2) we get

$$\psi_{k+1}(x) \leq c' \int_0^x dy \int_0^\ell e^{-\epsilon^{-1}\alpha|y-z|} \psi_k(z) \epsilon^{-1} dz \quad (\text{C.3})$$

Define $v_k(x) = e^{-bx}\psi_k(x)$, $b > 0$ a large constant whose value will be specified later. We have:

$$v_{k+1}(x) \leq c' \int_0^x e^{-b(x-y)} dy \int_0^\ell e^{-\epsilon^{-1}\alpha|y-z|+b(z-y)} v_k(z) \epsilon^{-1} dz \quad (\text{C.4})$$

For ϵ so small that $\epsilon^{-1}\alpha > b$ we have

$$\|v_{k+1}\| \leq c' \int_0^x e^{-b(x-y)} dy \frac{2\epsilon^{-1}}{\epsilon^{-1}\alpha - b} \|v_k\| \leq \frac{c'}{b} \frac{2\epsilon^{-1}}{\epsilon^{-1}\alpha - b} \|v_k\| \quad (\text{C.5})$$

We choose b so that $\frac{4c'}{\alpha b} = \frac{1}{2}$. Then for all ϵ so small that $\frac{\epsilon^{-1}}{\epsilon^{-1}\alpha - b} \leq \frac{2}{\alpha}$

$$\|v_{k+1}\| \leq \frac{1}{2} \|v_k\| \text{ which yields } \|\psi_{k+1}\| \leq e^{b\ell} \left(\frac{1}{2}\right)^k \|\psi_1\|.$$

APPENDIX D. SPECTRAL PROPERTIES OF $A_{h,m}$

In this appendix we shall first define a set \mathcal{A} by weakening properties of the pair (h_ϵ, m_ϵ) and then prove spectral properties of $A_{h,m}$ when (h, m) is in a small neighborhood of \mathcal{A} .

Instanton: additional notation

Referring to Appendix A for definition and properties of the instanton \bar{m} , we denote by \bar{m}_{x_0} , $x_0 \in (-1, 1)$, the translate of \bar{m} by $\epsilon^{-1}x_0$:

$$\bar{m}_{x_0}(x) = \bar{m}(x - \epsilon^{-1}x_0), \quad \bar{m}'_{x_0} = \frac{d\bar{m}_{x_0}}{dx}, \quad \bar{p}_{x_0}(x) = \beta(1 - \bar{m}_{x_0}(x)^2), \quad \bar{A}_{x_0} := \bar{p}_{x_0} J^* \quad (\text{D.1})$$

Properties of the pair (h_ϵ, m_ϵ)

- $m_\epsilon = \tanh\{\beta J^{\text{neum}} * m_\epsilon + h_\epsilon\}$, see (4.3).
- There are $r > 0$ and $b > 0$ so that $p_\epsilon(x) \leq e^{-b}$ for all $|x - \epsilon^{-1}x_0| \geq r$.
- $\|\frac{dm_\epsilon}{dx}\| < c'_{(4.4)}$ (proved in Lemma 4.1) and for any $c > 0$ there is $c' > 0$ so that for all ϵ small enough

$$\sup_{|x - \epsilon^{-1}x_0| \leq c \log \epsilon^{-1}} |m_\epsilon(x) - \bar{m}_{x_0}(x)| < c_{(\text{D.2})} \epsilon \log \epsilon^{-1} \quad (\text{D.2})$$

because by (3.15), $\|m^* - m_0\| \leq c\epsilon \log \epsilon^{-1}$.

- Since $\frac{dh_\epsilon}{dx} = \frac{-\epsilon j}{p_\epsilon(x)}$ and $\inf p_\epsilon > 0$ then $\|h_\epsilon\| \leq c_1$, $\|\frac{dh_\epsilon}{dx}\| < c\epsilon$ and, by (D.2),

$$\sup_{|x - \epsilon^{-1}x_0| \leq c \log \epsilon^{-1}} \left| \frac{dh_\epsilon(x)}{dx} - \frac{-\epsilon j}{\bar{p}_{x_0}(x)} \right| < c'_1 \epsilon^2 \log \epsilon^{-1}$$

The set \mathcal{A}

By default all coefficients a , c , C with or without a suffix are meant to be positive and independent of ϵ ; we shall indicate below by item n the n -th property of (h_ϵ, m_ϵ)

as listed in the previous paragraph and introduce the quantities (with b in (D.3) below the parameter entering in item 2)

$$C_{(\text{D.3})} > 1 : e^{-aC_{(\text{D.3})}(1-x_0)\log\epsilon^{-1}} = \epsilon^2, \quad a := \min\left\{\frac{b}{4}, a_{(\text{A.6})}, a_{(\text{A.4})}\right\} \quad (\text{D.3})$$

$$I = \{x : |x - \epsilon^{-1}x_0| \leq 2C_{(\text{D.3})}\log\epsilon^{-1}\}, \quad I' = \{x : |x - \epsilon^{-1}x_0| \leq C_{(\text{D.3})}\log\epsilon^{-1}\} \quad (\text{D.4})$$

(I' will be used later in Proposition D.1). With such notation we define \mathcal{A} as the collection of all pairs (h, m) such that $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and the following three inequalities hold:

$$p_{h,m}(x) = \beta(1 - m(x)^2) \leq e^{-2a_{(\text{D.5})}}, \quad |x - \epsilon^{-1}x_0| \geq r_{(\text{D.5})} \quad (\text{D.5})$$

$$\left\|\frac{dm}{dx}\right\| \leq C_{(\text{D.6})}, \quad \sup_{x \in I} |m(x) - \bar{m}_{x_0}(x)| \leq c'_{(\text{D.6})}\epsilon \log\epsilon^{-1} \quad (\text{D.6})$$

$$\|h\| \leq C_{(\text{D.7})}, \quad \left\|\frac{dh}{dx}\right\| \leq C_{(\text{D.7})}, \quad \sup_{x \in I} \left|\frac{dh(x)}{dx} - \frac{-\epsilon j}{\bar{p}_{x_0}(x)}\right| \leq c_{(\text{D.7})}\epsilon^2 \log\epsilon^{-1} \quad (\text{D.7})$$

where $C_{(\text{D.7})} > 2$ and:

- $r_{(\text{D.5})} > r$ and $2a_{(\text{D.5})} = b/2$, b and r are as in item 2.
- $C_{(\text{D.6})} > 2c'_{(\text{A.4})}$ and $c'_{(\text{D.6})} > 2c_{(\text{D.2})}$ (see item 3).
- $C_{(\text{D.7})} > 2\max\{c, c_1\}$ and $c_{(\text{D.7})} > 2c'_1$ (see item 4).

With the above choice of parameters $(h_\epsilon, m_\epsilon) \in \mathcal{A}$.

Spectral properties in a neighborhood of \mathcal{A}

We continue the analysis of the spectrum of $A_{h,m}$ started in Appendix A assuming that (h, m) is in the δ ball of \mathcal{A} defined as $\bigcup_{(h', m') \in \mathcal{A}} B_\delta(h, m)$, $B_\delta(h, m) := \{(h', m') : \|h - h'\| \leq \delta, \|m - m'\| \leq \delta\}$.

Proposition D.1. *There are positive constants $C_{(\text{D.8})}$, $c_{(\text{D.9})}$, $c'_{(\text{D.10})}$, $c_{(\text{D.11})}$ and $a_{(\text{D.11})}$ so that for any ϵ small enough there is $\delta = \delta(\epsilon) > 0$ and for any (h, m) in the δ ball of \mathcal{A}*

$$p_{h,m} \geq C_{(\text{D.8})} \quad (\text{D.8})$$

$$|\lambda_{h,m} - [1 - C_{(\text{D.9})}\epsilon]| \leq c_{(\text{D.9})}(\epsilon \log\epsilon^{-1})^2, \quad C_{(\text{D.9})} = |j| \frac{\langle \bar{m}' \rangle_\infty}{\langle (\bar{m}')^2 \rangle_\infty} > 0 \quad (\text{D.9})$$

Moreover, let $u_{h,m} > 0$ be normalized as $\langle u_{h,m}^2 \rangle_{h,m} = 1$ and I' is as in (D.4), then

$$\sup_{x \in I'} |u_{h,m}(x) - \tilde{m}'_{x_0}(x)| \leq c'_{(\text{D.10})}\epsilon(\log\epsilon^{-1})^2 \quad (\text{D.10})$$

$$u_{h,m}(x) \leq c_{(\text{D.11})}e^{-a_{(\text{D.5})}|x - \epsilon^{-1}x_0|} \quad (\text{D.11})$$

Proof. We shall first prove with slightly better coefficients the inequalities (D.8)–(D.11) when (h, m) is in \mathcal{A} and then use a continuity argument to extend the analysis to a δ ball of \mathcal{A} . We thus fix $(h, m) \in \mathcal{A}$ and drop the suffix (h, m) when no ambiguity may arise.

• Proof of (D.8). We bound $|m(x)| \leq \tanh\{\beta J^{\text{neum}} * \|m\| + \beta \|h\|\}$ and $\|h\| \leq C_{(\text{D.7})}$, hence $p_{h,m} \geq 2C_{(\text{D.8})}$, with $2C_{(\text{D.8})} = \beta(1 - s^2)$, s the positive solution of $s = \tanh\{\beta s + \beta C_{(\text{D.7})}\}$. (D.8) then follows in a δ ball of (h, m) if δ is small enough.

We shall next prove some rough bounds on λ and u which will then be improved as required in the proposition. We take here (h, m) in a δ -ball of \mathcal{A} with δ small enough. We are going to use repeatedly variants of the obvious equality:

$$\langle f A_{h,m} g \rangle_{h,m} = \langle f A_{h',m'} g \rangle_{h',m'} = \int f J^{\text{neum}} * g \quad (\text{D.12})$$

We have the lower bound $\lambda \geq \frac{\langle \bar{m}'_{x_0} A \bar{m}'_{x_0} \rangle_{h,m}}{\langle (\bar{m}'_{x_0})^2 \rangle_{h,m}}$, $A \equiv A_{h,m}$ and \bar{m}'_{x_0} here restricted to $\Lambda = \epsilon^{-1}[-1, 1]$. Using (D.12) we can rewrite the numerator as

$$\begin{aligned} \langle \bar{m}'_{x_0} A \bar{m}'_{x_0} \rangle_{h,m} &= \int_{\Lambda \times \Lambda} \bar{m}'_{x_0}(x) J^{\text{neum}}(x, y) \bar{m}'_{x_0}(y) = \int_{\mathbb{R} \times \mathbb{R}} \bar{m}'_{x_0}(x) J(x, y) \bar{m}'_{x_0}(y) + \Delta \\ &= \int_{\mathbb{R}} \bar{m}'_{x_0}(x)^2 / \bar{p}_{x_0} + \Delta = \int_{\Lambda} \bar{m}'_{x_0}(x)^2 / \bar{p}_{x_0} + \Delta' \\ &= \langle (\bar{m}'_{x_0})^2 \rangle_{h,m} + \int_{\Lambda} \bar{m}'_{x_0}(x)^2 \frac{p - \bar{p}_{x_0}}{p \bar{p}_{x_0}} + \Delta', \quad p \equiv p_{h,m} \end{aligned}$$

where by (A.4) and (D.3), $|\Delta|$ and $|\Delta'|$ are both bounded by $\leq c e^{-a_{(\text{A.4})} \epsilon^{-1}(1-x_0)} \leq c \epsilon^2$. The denominator in the last integral is bounded from below because $p \equiv p_{h,m} \geq C_{(\text{D.8})}$, ((D.8) has already been proved) and $\bar{p}_{x_0} \geq \beta(1 - m_\beta^2)$ (as $\bar{m}(x)$ converges monotonically to m_β as $x \rightarrow \infty$). By (D.6) and for δ small enough $|p(x) - \bar{p}_{x_0}(x)| \leq 2c'_{(\text{D.6})} \epsilon \log \epsilon^{-1}$ when $x \in I$, while in the complement we bound \bar{m}'_{x_0} as in (A.4) (recalling (D.3)) and use that $|p - \bar{p}_{x_0}| \leq \beta$. In conclusion we get

$$\lambda \geq 1 - c_{(\text{D.13})} \epsilon \log \epsilon^{-1} \quad (\text{D.13})$$

with $c_{(\text{D.13})}$ dependent on $C_{(\text{D.7})}, c'_{(\text{D.6})}, a_{(\text{D.5})}$.

• Proof of (D.11). We use (D.13) and the identity $u(x) = \lambda^{-n} (A^n u)(x)$ to get upper bounds on u . With $n = 1$ we obtain

$$\|u\| \leq \lambda^{-1} \|J\| \beta \sqrt{2} \left(\int u^2 \right)^{1/2} \leq \lambda^{-1} \|J\| \beta \sqrt{2} \left(\int \frac{\|p\|}{p} u^2 \right)^{1/2} \leq c \langle u^2 \rangle_{h,m}^{1/2} \quad (\text{D.14})$$

(having used Cauchy-Schwartz and that $\|p\| \leq \beta$). By tuning n with the distance from $\epsilon^{-1}x_0$ we get, using (D.5),

$$u(x) \leq [1 - c_{(\text{D.13})} \epsilon \log \epsilon^{-1}]^{-n} e^{-2a_{(\text{D.5})}n} \|u\|, \quad \text{when } |x - \epsilon^{-1}x_0| \geq n + r_{(\text{D.5})} \quad (\text{D.15})$$

which together with (D.14) proves (D.11) for $(h, m) \in \mathcal{A}$.

We shall next prove an upper bound on $\lambda \equiv \lambda_{h,m}$, (h, m) in a δ -ball of \mathcal{A} , δ suitably small. We start from the operator $\bar{A}_{x_0} = \bar{p}_{x_0} J^*$ acting on $L^\infty(\mathbb{R})$ and since 1 is its maximal eigenvalue (with eigenvector \bar{m}'_{x_0}), $1 \geq \frac{\langle u \bar{A}_{x_0} u \rangle_\infty}{\langle u^2 \rangle_\infty}$ where we choose

$u = u_{h,m}$ on $\Lambda = \epsilon^{-1}[-1, 1]$ and $u = 0$ on Λ^c . Denoting $\langle f \rangle_\infty = \int_{\mathbb{R}} \frac{f}{\bar{p}_{x_0}}$, we then

have

$$\langle u \bar{A}_{x_0} u \rangle_\infty = \int_{\Lambda \times \Lambda} u(x) J(x, y) u(y) = \int_{\Lambda \times \Lambda} u(x) J^{\text{neum}}(x, y) u(y) + R$$

with $R = - \int_{\Lambda \times \Lambda^c} u(x) J(x, y) u(y_\Lambda)$, y_Λ the reflection of y into Λ through its end-points. By (D.11) $|R| \leq c e^{-a_{(D.5)} |\epsilon^{-1}(1-x_0)|} \|u\|^2$, hence writing hereafter $\langle \cdot \rangle = \langle \cdot \rangle_{h,m}$,

$$\begin{aligned} \langle u^2 \rangle_\infty &\geq \langle u \bar{A}_{x_0} u \rangle_\infty \geq \langle u A u \rangle - c e^{-a_{(D.5)} |\epsilon^{-1}(1-x_0)|} \|u\|^2 \\ &= \lambda \langle u^2 \rangle - c e^{-a_{(D.5)} |\epsilon^{-1}(1-x_0)|} \|u\|^2 \end{aligned}$$

Thus, by (D.14), $\lambda \leq \frac{\langle u^2 \rangle_\infty}{\langle u^2 \rangle} + c e^{-a_{(D.5)} |\epsilon^{-1}(1-x_0)|}$. We postpone the proof that

$$\left| \frac{\langle u^2 \rangle}{\langle u^2 \rangle_\infty} - 1 \right| \leq c \epsilon^2 + c \epsilon \log \epsilon^{-1} \quad (\text{D.16})$$

and conclude, recalling (D.13) and pending the validity of (D.16),

$$1 - c_{(D.13)} \epsilon \log \epsilon^{-1} \leq \lambda \leq 1 + c_{(D.17)} \epsilon \log \epsilon^{-1} \quad (\text{D.17})$$

with $c_{(D.17)}$ dependent on $C_{(D.7)}$, $c'_{(D.6)}$, $a_{(D.5)}$.

Proof of (D.16)

Recalling that $p \geq C_{(D.8)}$, $p \leq \beta$, $\bar{p}_{x_0} \geq \beta(1 - m_\beta^2)$ and $\bar{p}_{x_0} \leq \beta$, we have

$$c_{(D.18)}^{-1} \leq \frac{\langle u^2 \rangle}{\langle u^2 \rangle_\infty} \leq c_{(D.18)} \quad (\text{D.18})$$

Then, by (D.11)

$$\frac{u(y)}{\langle u^2 \rangle_\infty^{1/2}} \leq c e^{-a_{(D.5)} |y - \epsilon^{-1} x_0|}, \quad y \in \Lambda \setminus I' \quad (\text{D.19})$$

hence by (D.3) and since $a_{(D.5)} = \frac{b}{4}$,

$$\int_{\Lambda \setminus I'} u^2 / \bar{p}_{x_0} \leq c_{(D.20)} \langle u^2 \rangle_\infty \epsilon^2, \quad \langle u^2 \rangle_\infty > \int_{I'} u^2 / \bar{p}_{x_0} \geq (1 - c_{(D.20)} \epsilon^2) \langle u^2 \rangle_\infty \quad (\text{D.20})$$

We also have

$$\begin{aligned} \left| \frac{\langle u^2 \rangle}{\langle u^2 \rangle_\infty} - \frac{\int_{I'} u^2 / p}{\langle u^2 \rangle_\infty} \right| &\leq c \epsilon^2, \quad \left| \frac{\langle u^2 \rangle}{\langle u^2 \rangle_\infty} - \frac{\int_{I'} u^2 / \bar{p}_{x_0}}{\langle u^2 \rangle_\infty} \right| \leq c \epsilon^2 + c \epsilon \log \epsilon^{-1} \\ \left| \frac{\langle u^2 \rangle}{\langle u^2 \rangle_\infty} - 1 \right| &\leq c \epsilon^2 + c \epsilon \log \epsilon^{-1} \end{aligned}$$

In the first inequality above we have used (D.19), in the second (D.6) and in the third (D.20). (D.16) is proved.

We shall next prove (D.10) that we split in an upper and a lower bound for $u = u_{h,m}$, we take here (h, m) in a δ -ball of \mathcal{A} with δ small enough

- Proof of (D.10) (the upper bound). Let $y \in I'$, then, writing below $y_0 \equiv y$,

$$\lambda^n u(y) = \int u(y_n) \prod_{k=1}^n \left\{ \bar{A}_{x_0}(y_{k-1}, y_k) \frac{p(y_{k-1})}{\bar{p}_{x_0}(y_{k-1})} \right\} dy_1 \cdots dy_n \quad (\text{D.21})$$

We choose again $n = C_{(\text{D.3})} \log \epsilon^{-1}$ observing that since $y_0 \in I'$ all y_k are in I . We bound $\lambda^{-n} \leq (1 - c_{(\text{D.13})} \epsilon \log \epsilon^{-1})^{-n} \leq (1 + n c \epsilon \log \epsilon^{-1}) \leq (1 + c' \epsilon [\log \epsilon^{-1}]^2)$. Since all y_k are in I , by (D.6) and for δ small enough,

$$\prod_{k=1}^n \frac{p(y_{k-1})}{\bar{p}_{x_0}(y_{k-1})} \leq 1 + c \epsilon [\log \epsilon^{-1}]^2$$

hence (with a new constant c)

$$u(y) \leq [1 + c \epsilon (\log \epsilon^{-1})^2] \int u(y_n) \prod_{k=1}^n \{ \bar{A}_{x_0}(y_{k-1}, y_k) \} dy_1 \cdots dy_n \quad (\text{D.22})$$

We define \tilde{u} so that $u(y_n) = \langle \tilde{m}'_{x_0} u \rangle_\infty \tilde{m}'_{x_0}(y_n) + \tilde{u}$. By (A.6)-(D.3) and for all $y \in I'$

$$u(y) \leq \tilde{m}'_{x_0}(y) [1 + c \epsilon (\log \epsilon^{-1})^2] \langle \tilde{m}'_{x_0} u \rangle_\infty + c \epsilon^2 \|u\| \quad (\text{D.23})$$

which by (D.14) can be rewritten as

$$\frac{u(y)}{\langle u^2 \rangle^{1/2}} \leq \{ [1 + c \epsilon (\log \epsilon^{-1})^2] \frac{\langle \tilde{m}'_{x_0} u \rangle_\infty}{\langle u^2 \rangle^{1/2}} \} \tilde{m}'_{x_0}(y) + c \epsilon^2 \quad (\text{D.24})$$

By Cauchy-Schwartz,

$$\frac{u(y)}{\langle u^2 \rangle^{1/2}} \leq \left(\frac{\langle u^2 \rangle_\infty}{\langle u^2 \rangle} \right)^{1/2} \left\{ [1 + c \epsilon (\log \epsilon^{-1})^2] \tilde{m}'_{x_0}(y) \right\} + c \epsilon^2 \quad (\text{D.25})$$

which, by (D.16), proves

$$\frac{u(x)}{\langle u^2 \rangle^{1/2}} \leq \tilde{m}'_{x_0}(x) + \frac{c'_{(\text{D.10})}}{2} \epsilon (\log \epsilon^{-1})^2 \quad (\text{D.26})$$

- Proof of (D.10) (the lower bound). Proceeding in a similar way we get the lower bound:

$$\frac{u(y)}{\langle u^2 \rangle^{1/2}} \geq \{ [1 - c \epsilon (\log \epsilon^{-1})^2] \frac{\langle \tilde{m}'_{x_0} u \rangle_\infty}{\langle u^2 \rangle^{1/2}} \} \tilde{m}'_{x_0}(y) - c \epsilon^2 \quad (\text{D.27})$$

To bound the curly bracket from below we multiply both sides of (D.23) by $\bar{p}_{x_0}^{-1} u$ and integrate over I' . By (D.20):

$$(1 - c_{(\text{D.20})} \epsilon^2) \langle u^2 \rangle_\infty \leq \langle \tilde{m}'_{x_0} u \rangle_\infty^2 [1 + c \epsilon (\log \epsilon^{-1})^2] + c \epsilon^2 \log \epsilon^{-1} \|u\|^2$$

By (D.14), $(1 - c_{(\text{D.20})} \epsilon^2) \leq \frac{\langle \tilde{m}'_{x_0} u \rangle_\infty^2}{\langle u^2 \rangle_\infty} [1 + c \epsilon (\log \epsilon^{-1})^2] + c \epsilon^2 \log \epsilon^{-1}$, hence

$$1 - c \epsilon (\log \epsilon^{-1})^2 \leq \frac{\langle \tilde{m}'_{x_0} u \rangle_\infty^2}{\langle u^2 \rangle_\infty} \leq 1 \quad (\text{D.28})$$

which by (D.27) yields $\frac{u(y)}{\langle u^2 \rangle^{1/2}} \geq \left(\frac{\langle u^2 \rangle_\infty}{\langle u^2 \rangle} \right)^{1/2} [1 - c \epsilon (\log \epsilon^{-1})^2] \tilde{m}'_{x_0}(y) - c \epsilon (\log \epsilon^{-1})^2$.

Using (D.16) we then get

$$u_{h,m}(x) \geq \tilde{m}'_{x_0}(x) - \frac{c'_{(\text{D.10})}}{2} \epsilon (\log \epsilon^{-1})^2 \quad (\text{D.29})$$

• Proof of (D.9). We first suppose $(h, m) \in \mathcal{A}$ and use for the first time the conditions on dm/dx and dh/dx contained in the definition of \mathcal{A} . Writing f' for the derivative of f w.r.t. x , we differentiate $m(x) = \tanh\{\beta J^{\text{neum}} * m(x) + \beta h(x)\}$ and get $m' = pJ^{\text{neum}} * m' + ph'$, hence $Lm' = -ph'$, $L = A - 1$. We multiply both sides by $p^{-1}u$ and integrate over x . Recalling that L is selfadjoint in the scalar product with weight p^{-1} , we then have

$$(\lambda - 1)\langle um' \rangle = -\langle uph' \rangle \quad (\text{D.30})$$

By (D.11), $|\langle um' \rangle - \int_{I'} p^{-1}um'| \leq c\epsilon^2$, having used that $|m'|$ is bounded, first inequality in (D.6). Since $m' = pJ^{\text{neum}} * m' + ph'$, using the second inequality in (D.7),

$$|m'(x) - p(J^{\text{neum}})' * m(x)| \leq \sup_{y \in I'} |ph'| \leq c\epsilon, \quad x \in I'$$

Then, by the second inequality in (D.6),

$$|m'(x) - pJ^{\text{neum}} * \bar{m}'(x)| = |m'(x) - p(J^{\text{neum}})' * \bar{m}(x)| \leq c\epsilon \log \epsilon^{-1}, \quad x \in I'$$

and by (D.10) and (D.11),

$$|\langle um' \rangle - \langle \bar{m}' \bar{m} \rangle_\infty| \leq c\epsilon \log \epsilon^{-1} \quad (\text{D.31})$$

Analogous estimates hold for $\langle uph' \rangle$ and we get

$$|\lambda - [1 - C_{(\text{D.9})}\epsilon]| \leq \frac{c_{(\text{D.9})}}{2}(\epsilon \log \epsilon^{-1})^2 \quad (\text{D.32})$$

To conclude the proof of the Proposition we need to extend the previous bounds to (\hat{h}, \hat{m}) in a δ -ball around (h, m) . By (D.12)

$$\frac{\hat{\lambda}}{\lambda} \geq \frac{\langle u^2 \rangle}{\langle u^2 \rangle_{\hat{h}, \hat{m}}} \geq c\delta \quad (\text{D.33})$$

The analogous bound can be proved for $\lambda/\hat{\lambda}$ and (D.9) follows if δ is small enough. The proof of Proposition D.1 is complete. \square

The rest of the spectrum is separated from $\lambda_{h,m}$ by a spectral gap, see [4].

Proposition D.2. *There are $c_{(\text{D.34})}, a_{(\text{D.34})} > 0$, $c_{(\text{D.35})}$ and $a_{(\text{D.35})} > 0$ so that for all ϵ small enough the following holds. For any $(h', m') \in \mathcal{A}$ there is $\delta = \delta(\epsilon)$ so that for all (h, m) in a δ -ball around (h', m') , for all bounded ψ*

$$\|A_{h,m}^n \tilde{\psi}\| \leq c_{(\text{D.34})} e^{-a_{(\text{D.34})}n} \|\psi\|, \quad \tilde{\psi} = \psi - \frac{\langle \psi u_{h,m} \rangle_{h,m}}{\langle u_{h,m}^2 \rangle_{h,m}} u_{h,m} \quad (\text{D.34})$$

$$|L_{h,m}^{-1} \tilde{\psi}(x)| \leq c_{(\text{D.35})} \int e^{-a_{(\text{D.35})}|x-y|} |\tilde{\psi}(y)| dy \quad (\text{D.35})$$

The operator A^ and its spectral properties.*

We conclude this appendix with a simple extension of the previous results which will allow us to complete the proof of Theorem 2.1 and of Lemma 4.1. Let (h^*, m^*) be the solution of the antisymmetric problem in $\epsilon^{-1}[-1, \ell^*]$, with x_0 the middle point in $[-1, \ell^*]$. We denote by A^* the operator $p^* J^{\text{neum},*}$ acting on $L^\infty(\epsilon^{-1}[-1, \ell^*])$

with $p^* = p_{h^*, m^*}$ and kernel $J^{\text{neum},*}(x, y)$ (defined with Neumann conditions on $\epsilon^{-1}[-1, \ell^*]$). We denote by $\langle \cdot \rangle_*$ the integral over $\epsilon^{-1}[-1, \ell^*]$ w.r.t. the measure $(p^*)^{-1}dx$. We first observe that the pair (h^*, m^*) satisfies the same properties (with same parameters) as the pair (h_ϵ, m_ϵ) (recall that m_ϵ is the restriction of m^* to $\epsilon^{-1}[-1, 1]$ and that h_ϵ is the restriction of h^* except for the additive term R_ϵ). It then follows that λ^* and u^* satisfy the same properties as $\lambda_{h, m}$ and $u_{h, m}$ stated in Proposition D.1 (without loss of generality we may suppose with same coefficients). Also Proposition D.2 remains valid, indeed its validity is quite general as discussed in Section 8.3 of [9].

Conclusion of the proof of Theorem 2.1. In order to keep the notation used so far we replace the original interval $\epsilon^{-1}[-\ell, \ell]$ in Theorem 2.1 by the interval $\epsilon^{-1}[-1, \ell^*]$ and denote the solution (h_ϵ, m_ϵ) of Theorem 2.1 by (h^*, m^*) . Reminding that it only remains to prove that $m^*(x)$ is an increasing function of x (we are supposing $j < 0$), we shorthand $\psi = \frac{dm^*}{dx}$ and shall prove that $\psi(x)$ is strictly positive at all x . We have

$$\psi = L^{-1} \left(-p^* \frac{dh^*}{dx} \right) = L^{-1}(\epsilon j) \quad (\text{D.36})$$

where $L = A^* - 1$. The positivity of ψ then follows from

$$L^{-1}(\epsilon j) = \sum_{n=0}^{\infty} (A^*)^n (-\epsilon j) \quad (\text{D.37})$$

once we prove that the series converges (as all its elements are positive). Convergence follows because there are $a = a(\epsilon)$ and $c = c(\epsilon)$ positive such that for all n ,

$$\|(A^*)^n\| \leq ce^{-an} \quad (\text{D.38})$$

which would be easy if this was the L^2 norm as we know that λ^* is the maximal eigenvalue and $\lambda^* < 1 - c\epsilon$.

• Proof of (D.38). With λ^* and u^* the maximal eigenvalue and eigenvector of A^* , u^* normalized, $\langle (u^*)^2 \rangle_* = 1$, we have

$$(A^*)^n \psi = (\lambda^*)^n \langle u^* \psi \rangle_* u^* + (A^*)^n \tilde{\psi}, \quad \tilde{\psi} = \psi - \langle u^* \psi \rangle_* u^* \quad (\text{D.39})$$

We have $\lambda^* < 1 - C\epsilon$, $C > 0$, (by (D.9)), we bound u^* using (D.11), then by (D.34)

$$\|(A^*)^n \psi\| \leq c(\lambda^*)^n \|\psi\| + c_{(\text{D.34})} e^{-a_{(\text{D.34})} n} \|\psi\| \quad (\text{D.40})$$

hence (D.38). \square

Conclusion of the proof of Lemma 4.1. It only remains to prove the second inequality in (4.4). With $\psi = \frac{dm^*}{dx}$, by (D.36), and using the previous notation

$$\psi = \left([\lambda^* - 1]^{-1} \epsilon j \int_{-\epsilon^{-1}}^{\epsilon^{-1} \ell^*} u^* \right) u^* + L^{-1} \phi, \quad \phi = (\epsilon j) - (\epsilon j \int u^*) u^* \quad (\text{D.41})$$

$|\lambda^* - 1|^{-1} \epsilon j| \leq c$ by (D.9) and by (D.11):

$$\int u^* \leq c, \quad \sup_{|x - \epsilon^{-1} x_0| \geq r_{(4.4)} \log \epsilon^{-1}} u^*(x) \leq c_{(\text{D.11})} e^{-a_{(\text{D.5})} r_{(4.4)} \log \epsilon^{-1}}$$

By choosing $a_{(D.5)}r_{(4.4)} > 1$ the first term on the r.h.s. of (D.41) is bounded by $c\epsilon$ when $|x - \epsilon^{-1}x_0| \geq r_{(4.4)} \log \epsilon^{-1}$. The last term is bounded using (D.34) by

$$\leq c_{(D.34)} \sum_{n=0}^{\infty} e^{-a_{(D.34)}n} \|\phi\| \leq c'\epsilon$$

because $\|\phi\| \leq c\epsilon$. Lemma 4.1 is proved. \square

APPENDIX E. AN AUXILIARY DYNAMICS

We return in this appendix to the analysis of the auxiliary dynamics introduced in Appendix A. We shall study the case where initially $(h_0, m_0) \in \mathcal{A}$ and prove a local existence and uniqueness theorem under suitable assumptions on $h(t)$. We would like to work in \mathcal{A} but \mathcal{A} itself is not nice in the L^∞ topology we are using as it involves derivatives. For this reason we introduced the δ -balls of \mathcal{A} in the previous appendix which will play an important role here as well. Our first result is a straight consequence of Proposition D.1 and Proposition D.2 and its proof is omitted:

Proposition E.1. *There is $c > 0$ and for any $\epsilon > 0$ small enough there is $\delta = \delta(\epsilon) > 0$ not larger than the parameter δ in Proposition D.1 so that for any $(h_0, m_0) \in \mathcal{A}$ and any (h', m') and (h'', m'') in the δ -ball of (h_0, m_0)*

$$\|L_{h', m'}^{-1}\| \leq c\epsilon^{-1}, \quad \|L_{h', m'}^{-1} - L_{h'', m''}^{-1}\| \leq c\epsilon^{-2}(\|h' - h''\| + \|m' - m''\|) \quad (\text{E.1})$$

Proposition E.2. *Let δ and c be as in Proposition E.1 and let C be any positive number. Then for any $\epsilon > 0$ small enough there is $T \in (0, \frac{\delta}{2C})$ so that the following holds. For any $(h_0, m_0) \in \mathcal{A}$, and any $h(t)$, $t \in [0, T]$, such that $h(0) = h_0$ and $\|\frac{dh(t)}{dt}\| \leq C$ there is $m(t)$, $t \in [0, T]$, such that:*

$$\frac{dm}{dt} = L_t^{-1}(-p_t \frac{dh}{dt}), \quad m(0) = m_0, \quad L_t = L_{h(t), m(t)}, \quad p_t = p_{h(t), m(t)} \quad (\text{E.2})$$

*($L_t = L_{h(t), m(t)}$, $p_t = p_{h(t), m(t)}$), $\|m(\cdot) - m_0\| \leq \frac{\delta}{2}$ and $m(\cdot) = \tanh\{\beta J^{\text{neum}} * m(\cdot) + \beta h(\cdot)\}$. Finally $m(\cdot)$ is the unique solution of (E.2) in $\|m(\cdot) - m_0\| \leq \frac{\delta}{2}$.*

Proof. T is determined by the following three conditions:

$$T < \frac{\delta}{2C}, \quad c\epsilon^{-1}\beta CT < \frac{\delta}{2}, \quad (c\epsilon^{-1}2\beta^2 + \beta c\epsilon^{-2})T < 1 \quad (\text{E.3})$$

The first one ensures that $\|h(\cdot) - h_0\| < \frac{\delta}{2}$ (because $\|\frac{dh(t)}{dt}\| \leq C$); the second one (obtained by bounding $L_t^{-1}(-p_t \frac{dh}{dt})$ via Proposition E.1) will imply that $\|m(\cdot) - m_0\| < \frac{\delta}{2}$, so that $(h(t), m(t))$ is always in the δ -ball of (h_0, m_0) and Proposition E.1 can be applied. The third condition will imply that the integral version of (E.2) gives rise to a contraction.

Let $\mathcal{X} := \{m \in C([0, T], L^\infty(\epsilon^{-1}[-1, 1]; [-1, 1])) : m(0) = m_0, \|m(\cdot) - m_0\| \leq \frac{\delta}{2}\}$ and for $m \in \mathcal{X}$ let

$$\psi(m)(t) = m_0 + \int_0^t L_s^{-1} \left(-p_s \frac{dh(s)}{ds} \right) \quad (\text{E.4})$$

By (E.1) and the second inequality in (E.3), $\|\psi(m)(t) - m_0\| \leq c\epsilon^{-1}\beta Ct < \frac{\delta}{2}$. Thus ψ maps \mathcal{X} into itself. By (E.1) and the third inequality in (E.3) ψ is a contraction with sup norm in x and t . Therefore there is a fixed point $m \in \mathcal{X}$: $m = \psi(m)$ and since ψ maps \mathcal{X} into functions which are differentiable in t with bounded derivative, m is a solution of (E.2). By (E.2)

$$\frac{d}{dt} \left(m(t) - \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\} \right) = 0$$

so that $m(t) - \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\} = m_0 - \tanh\{\beta J^{\text{neum}} * m_0 + \beta h_0\} = 0$. Finally, if m solves (E.2) and $\|m(\cdot) - m_0\| \leq \frac{\delta}{2}$, then $m \in \mathcal{X}$ and $\psi(m) = m$. Since ψ is a contraction m is unique. \square

APPENDIX F. PROPERTIES OF THE SETS \mathcal{A} AND \mathcal{G}

The intervals I and I' which appear frequently in the sequel have been defined in (D.4).

Fixing the parameters in the set \mathcal{G}

The coefficient $a_{(4.6)}$ is a positive number strictly smaller than all the parameters in Appendix D involved with exponential decay, in particular we require $a_{(4.6)} < \min\{a_{(D.5)}(1 - x_0), a_{(D.35)} \frac{1+x_0}{1-x_0}\}$. The other parameter $b_{(4.7)}$ is fixed so that:

$$b_{(4.7)} < 1 \text{ and such that } \left\{ \frac{3\beta c_{(D.35)}}{a_{(D.35)} - a_{(4.6)} \frac{1-x_0}{1+x_0}} \right\} b_{(4.7)} < \frac{e^{-2a_{(D.5)}} - e^{-4a_{(D.5)}}}{2\beta} \quad (\text{F.1})$$

Proposition F.1. *For all $\epsilon > 0$ small enough if $h \in \mathcal{G}$ there is m such that $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and $(h, m) \in \mathcal{A}$.*

Proof. Given $h \in \mathcal{G}$ and $t \in [0, 1]$ we define $h(t) := th + (1-t)h_\epsilon$ observing that $(h(0), m(0)) := (h_\epsilon, m_\epsilon) \in \mathcal{A}$ by the definition of \mathcal{A} . Let S be the sup of all $s \leq 1$ such that there exists $m(t)$, $t \in [0, s]$, which solves (E.2) in $[0, s]$ starting from $m(0) = m_\epsilon$ and such that for all such t , $(h(t), m(t))$ is in the δ -ball of \mathcal{A} with δ as Proposition D.1. We shall prove that $S = 1$ and that for all $t \leq 1$ $(h(t), m(t)) \in \mathcal{A}$ thus proving the Proposition.

Since $\|\frac{dh(t)}{dt}\| = \|h - h_\epsilon\| \leq b_{(4.7)}$ (because $h \in \mathcal{G}$) we can apply Proposition E.2 with $C = b_{(4.7)}$ and $(h_0, m_0) = (h_\epsilon, m_\epsilon) \in \mathcal{A}$. As a consequence there is $T = T(\epsilon) > 0$ so that $m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$, $t \in [0, T]$, and $\delta = \delta(\epsilon)$ so that $\|h(t) - h_\epsilon\| \leq \delta/2$, $\|m(t) - m_\epsilon\| \leq \delta/2$, $t \in [0, T]$. Since δ is not larger than the

parameter δ of Proposition D.1 (see Proposition E.1) we then conclude that $S \geq T$. By the definition of S the bounds in Proposition D.1 hold for $(h(t), m(t))$ at any $t \in [0, S]$ and it is now just a matter of computations to check that $(h(t), m(t)) \in \mathcal{A}$ for all such t . We start by proving that $h(t)$ satisfies the conditions in (D.7).

$$\|h(t)\| \leq \|h_\epsilon\| + \|h - h_\epsilon\| \leq \frac{C_{(D.7)}}{2} + b_{(4.7)} \leq C_{(D.7)}, \quad (\text{as } b_{(4.7)} < 1 < \frac{C_{(D.7)}}{2})$$

$$\left\| \frac{dh(t)}{dx} \right\| \leq \left\| \frac{dh_\epsilon}{dx} \right\| + \left\| \frac{d(h - h_\epsilon)}{dx} \right\| \leq \frac{C_{(D.7)}}{2} + \epsilon \leq C_{(D.7)}$$

for ϵ small enough. Finally in I (defined in (D.4))

$$\left| \frac{dh(t)}{dx} - \frac{-\epsilon j}{p_{x_0}} \right| \leq \left| \frac{dh_\epsilon}{dx} - \frac{-\epsilon j}{p_{x_0}} \right| + t \left| \frac{d(h - h_\epsilon)}{dx} \right| \leq \frac{c_{(D.7)}}{2} \epsilon^2 \log \epsilon^{-1} + \epsilon^2 \leq c_{(D.7)} \epsilon^2 \log \epsilon^{-1}$$

(for ϵ small enough) so that also the last condition in (D.7) is satisfied.

We shall next prove that $m(t)$ satisfies the conditions required in \mathcal{A} . We write $f(t) := -p_t[h - h_\epsilon]$; $\lambda_t, u(t)$ for the maximal eigenvalue and eigenvector of A_t ; $\langle \cdot \rangle_t$ for the integral of the measure $p_t^{-1} dx$ on $\epsilon^{-1}[-1, 1]$; $\tilde{f}(t) := f(t) - \langle u(t)f(t) \rangle_t u(t)$. By (E.2)

$$\frac{dm(t)}{dt} = L_t^{-1} f(t) = \lambda_t^{-1} \langle u(t)f(t) \rangle_t u(t) + L_t^{-1} \tilde{f}(t), \quad \langle u(t)^2 \rangle_t = 1 \quad (\text{F.2})$$

We bound $|f(t)| \leq \beta N(h - h_\epsilon) E_\epsilon(x)^{-1}$ (using that $h \in \mathcal{G}$ and $p_t \leq \beta$), $u(t) \leq c_{(D.11)} e^{-a_{(D.5)} |x - \epsilon^{-1} x_0|}$ and get

$$\begin{aligned} |\langle u(t)f(t) \rangle_t| &\leq cN(h - h_\epsilon) e^{-a_{(4.6)}(1-x_0)\epsilon^{-1}} \\ |\tilde{f}(t)| &\leq N(h - h_\epsilon) \left(c e^{-a_{(4.6)}(1-x_0)\epsilon^{-1}} c_{(D.11)} e^{-a_{(D.5)} |x - \epsilon^{-1} x_0|} + \beta E_\epsilon^{-1} \right) \end{aligned} \quad (\text{F.3})$$

By (D.35) $|L_t^{-1} \tilde{f}(t)|(x) \leq c_{(D.35)} \int e^{-a_{(D.35)} |x-y|} |\tilde{f}(t)| dy$ and

$$\int e^{-a_{(D.35)} |x-y|} E_\epsilon(y)^{-1} \leq \frac{2E_\epsilon(x)^{-1}}{a_{(D.35)} - a_{(4.6)}}$$

so that, by (F.2) and (F.3) and since $\lambda_t \leq c\epsilon^{-1}$

$$\begin{aligned} |m(t) - m_\epsilon| &\leq N(h - h_\epsilon) \left(c' \epsilon^{-1} e^{-a_{(4.6)}(1-x_0)\epsilon^{-1}} c_{(D.11)} e^{-a_{(D.5)} |x - \epsilon^{-1} x_0|} \right. \\ &\quad \left. + \frac{2\beta c_{(D.35)} E_\epsilon(x)^{-1}}{a_{(D.35)} - a_{(4.6)}^-} \right) \end{aligned} \quad (\text{F.4})$$

which (recalling that $N(h - h_\epsilon) \leq b_{(4.7)}$) proves that for ϵ small enough,

$$\|m(t) - m_\epsilon\| \leq \kappa := \frac{3\beta c_{(D.35)} b_{(4.7)}}{a_{(D.35)} - a_{(4.6)}^-} \quad (\text{F.5})$$

$$N(m(t) - m_\epsilon) \leq \left(c' \epsilon^{-1} c_{(D.11)} + \frac{2\beta c_{(D.35)}}{a_{(D.35)} - a_{(4.6)}^-} \right) N(h - h_\epsilon) \quad (\text{F.6})$$

By (F.5), $p_t \leq \beta(1 - (|m_\epsilon| - \kappa)^2) < e^{-4a_{(D.5)}} + 2\beta\kappa$ in $\{|x - \epsilon^{-1} x_0| \geq r_{(D.5)}\}$ and therefore, by (F.1), $m(t)$ satisfies (D.5) for $t \in [0, T]$. To prove the second condition

in (D.6) we write for $x \in I$,

$$\begin{aligned} |m(x, t) - \bar{m}_{x_0}(x)| &\leq |m(x, t) - m_\epsilon(x)| + |m_\epsilon(x) - \bar{m}_{x_0}(x)| \\ &\leq N(m(t) - m_\epsilon)E_\epsilon(x)^{-1} + c_{(D.2)}\epsilon \log \epsilon^{-1} \\ &\leq c\epsilon^{-1}E_\epsilon(x)^{-1} + c_{(D.2)}\epsilon \log \epsilon^{-1} \leq 2c_{(D.2)}\epsilon \log \epsilon^{-1} \end{aligned}$$

for ϵ small enough (because $E_\epsilon(x)^{-1} \leq e^{-a_{(4.6)}[\epsilon^{-1}(1-x_0)-2C_{(D.3)}\log \epsilon^{-1}]}$). The second inequality in (D.6) then follows recalling that $c'_{(D.6)} > 2c_{(D.2)}$. To prove the first inequality in (D.6) we take the x -derivative of the equality $m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$:

$$\begin{aligned} \frac{dm(t)}{dx} &= \lambda_t^{-1} \langle u(t)g(t) \rangle_t u(t) + L_t^{-1} \tilde{g}(t) \\ g(t) &:= \{-p_t \frac{d[h - h_\epsilon]}{dx}\}, \quad \tilde{g}(t) = g(t) - \langle u(t)g(t) \rangle_t u(t) \end{aligned} \quad (\text{F.7})$$

By (4.7), $\|g(t)\| \leq \beta\epsilon$ and an argument similar to the previous one shows that $\|\frac{dm(t)}{dx}\| \leq c\epsilon$, so that also the first condition in (D.6) is satisfied.

In conclusion we have proved so far that for all ϵ small enough, $(h(t), m(t)) \in \mathcal{A}$ for all $t \in [0, S]$. Suppose by contradiction that $S < 1$, write $S' = \min\{1, S + T\}$, then since $(h(S), m(S)) \in \mathcal{A}$ by Proposition E.2 there is $m(t)$, $t \in [S, S']$, which solves (E.2) in $[S, S']$ starting from $m(S)$ and such that for all such t , $(h(t), m(t))$ is in the δ -ball of $(h(S), m(S))$, hence, a fortiori, in the δ -ball of \mathcal{A} with δ as in Proposition D.1. This contradicts the maximality of S hence $S = 1$. \square

Proposition F.2. *There are $a_{(F.8)} > 0$, $r_{(F.8)} > 0$, c, c' and $a_{(F.9)} > 0$ so that for all ϵ small enough the following holds. Let $h \in \mathcal{G}$ and $(h, m) \in \mathcal{A}$ (existence of m follows from Proposition F.1), then*

$$\sup_{|x - \epsilon^{-1}x_0| \leq 2r_{(F.8)}\epsilon^{-1}} |m(x) - m_\epsilon(x)| \leq e^{-a_{(F.8)}\epsilon^{-1}} \quad (\text{F.8})$$

$$|\lambda_{h,m} - \lambda_\epsilon| \leq ce^{-a_{(F.8)}\epsilon^{-1}}, \quad \|u_{h,m} - u_\epsilon\| \leq c'e^{-a_{(F.9)}\epsilon^{-1}} \quad (\text{F.9})$$

Proof. (F.8) follows from (F.6), in the sequel it is convenient to have $a_{(F.8)}$ small, in particular $a_{(F.8)} < a_{(D.11)}$. Let λ and u be the maximal eigenvalue and eigenvector of $A := A_{h,m}$, $u > 0$ normalized so that $\langle u^2 \rangle = 1$ ($\langle \cdot \rangle := \langle \cdot \rangle_{h,m}$). and $\lambda_\epsilon, u_\epsilon$ the maximal eigenvalue and eigenvector of $A_\epsilon := A_{h_\epsilon, m_\epsilon}$ with $u_\epsilon > 0$ normalized so that $\langle u_\epsilon^2 \rangle_\epsilon = 1$ ($\langle \cdot \rangle_\epsilon := \langle \cdot \rangle_{h_\epsilon, m_\epsilon}$). Since (h_ϵ, m_ϵ) and (h, m) are both in \mathcal{A} we can use the bounds established in Proposition D.1 and D.2 for A and A_ϵ .

We then have

$$\frac{\lambda}{\lambda_\epsilon} \geq \frac{\langle u_\epsilon^2 \rangle_\epsilon}{\langle u_\epsilon^2 \rangle} = 1 - \frac{\langle u_\epsilon^2(1 - \frac{p}{p_\epsilon}) \rangle}{\langle u_\epsilon^2 \rangle} \geq 1 - ce^{-a_{(F.8)}\epsilon^{-1}} \quad (\text{F.10})$$

the first inequality following from (D.12). To prove the last one we recall that $p_\epsilon = p_{h^*, m^*} \geq c_{(4.1)}$, $p \equiv p_{h,m} \geq C_{(D.8)}$. In $\{x : |x - \epsilon^{-1}x_0| \leq 2r_{(F.8)}\epsilon^{-1}\}$ we use (F.8) to get

$$\sup_{|x - \epsilon^{-1}x_0| \leq 2r_{(F.8)}\epsilon^{-1}} \left|1 - \frac{p}{p_\epsilon}\right| \leq ce^{-a_{(F.8)}\epsilon^{-1}} \quad (\text{F.11})$$

In $\{x : |x - \epsilon^{-1}x_0| > 2r_{(F.8)}\epsilon^{-1}\}$ we bound $|1 - \frac{p}{p_\epsilon}| \leq \frac{2\beta}{c_{(4.1)}C_{(D.8)}}$ and u_ϵ using (D.11). Same argument is used to bound from below $\frac{\lambda_\epsilon}{\lambda}$ and the first inequality in (F.9) follows because λ and λ_ϵ are both close to 1 by $c\epsilon$.

In order to compute the sup in the second inequality in (F.9) we consider first $|x - \epsilon^{-1}x_0| > r_{(F.8)}\epsilon^{-1}$. In such a case both u and u_ϵ are smaller than $c_{(D.11)}e^{-a_{(D.5)}\epsilon^{-1}r_{(F.8)}}$ hence their difference is bounded by $c'e^{-a_{(F.9)}\epsilon^{-1}}$, provided $a_{(F.9)} < a_{(D.5)}r_{(F.8)}$. We next take $|x - \epsilon^{-1}x_0| \leq r_{(F.8)}\epsilon^{-1}$. Analogously to (D.21) and with $y_0 \equiv x$,

$$\lambda^N u(x) = \int u(y_N) \prod_{k=1}^N \{A_\epsilon(y_{k-1}, y_k) \frac{p(y_{k-1})}{p_\epsilon(y_{k-1})}\} dy_1 \cdots dy_N \quad (F.12)$$

We choose $N = b\epsilon^{-1}$ with $b > 0$ smaller than $r_{(F.8)}$. Then for all $k \leq N$, $|y_k - \epsilon^{-1}x_0| \leq (r_{(F.8)} + b)\epsilon^{-1} \leq 2r_{(F.8)}\epsilon^{-1}$ so that by (F.11) for all ϵ small enough, $u(x) \leq \lambda^{-N} [1 + ce^{-a_{(F.8)}\epsilon^{-1}}]^N A_\epsilon^N u(x)$. We then write $A_\epsilon^N u(x) = \lambda_\epsilon^N \langle uu_\epsilon \rangle_\epsilon u_\epsilon + A_\epsilon^N \tilde{u}(x)$ so that in $\{|x - \epsilon^{-1}x_0| \leq r_{(F.8)}\epsilon^{-1}\}$

$$u \leq [1 + ce^{-a_{(F.8)}\epsilon^{-1}}]^N \left(\left(\frac{\lambda_\epsilon}{\lambda} \right)^N \langle uu_\epsilon \rangle_\epsilon u_\epsilon + \lambda^{-N} \|u\| c_{(D.34)} e^{-a_{(D.34)}N} \right)$$

By (D.14)

$$u \leq [1 + ce^{-a_{(F.8)}\epsilon^{-1}}]^N \left(\left(\frac{\lambda_\epsilon}{\lambda} \right)^N \langle uu_\epsilon \rangle_\epsilon u_\epsilon + \lambda^{-N} ce^{-a_{(D.34)}N} \right) \quad (F.13)$$

We bound $\langle uu_\epsilon \rangle_\epsilon \leq \langle u^2 \rangle_\epsilon^{1/2}$, $\langle u^2 \rangle_\epsilon = 1 - \langle u^2 | 1 - \frac{p}{p_\epsilon} | \rangle$ and use the previous bounds for $|1 - \frac{p}{p_\epsilon}|$ so that $\langle uu_\epsilon \rangle_\epsilon \leq 1 + ce^{-a\epsilon^{-1}}$ with a and c suitable positive constants. By (F.10)

$$\left(\frac{\lambda_\epsilon}{\lambda} \right)^N \leq e^{-N \log\{1 - ce^{-a_{(F.8)}\epsilon^{-1}}\}} \leq \exp\{c'\epsilon^{-1}e^{a_{(F.8)}\epsilon^{-1}}\} \leq 1 + c''\epsilon^{-1}e^{a_{(F.8)}\epsilon^{-1}}$$

Collecting all these bounds and recalling that $\lambda < 1 - c\epsilon$, we get from (F.13)

$$\begin{aligned} u(x) &\leq (1 + c\epsilon^{-1}e^{-a_{(F.8)}\epsilon^{-1}}) u_\epsilon(x) \\ &\quad + c\epsilon^{-N} (\log(1 - c\epsilon) - a_{(D.34)}) \end{aligned} \quad (F.14)$$

hence the upper bound for u in (F.9). The lower bound is proved similarly. \square

Recall that (h^*, m^*) is the solution of the antisymmetric problem in $\epsilon^{-1}[-1, \ell^*]$, with x_0 the middle point in $[-1, \ell^*]$ and m_ϵ the restriction of m^* to $\epsilon^{-1}[-1, 1]$. We denote by λ^* and u^* the maximal eigenvalue and eigenvector of A_{h^*, m^*} and by λ_ϵ and u_ϵ those of $A_\epsilon = A_{h_\epsilon, m_\epsilon}$, writing u_ϵ also for its extension to $\epsilon^{-1}[-1, \ell^*]$ with $u_\epsilon = 0$ outside $\epsilon^{-1}[-1, 1]$. We suppose $\langle u_\epsilon^2 \rangle_\epsilon = \langle (u^*)^2 \rangle_* = 1$ with the obvious meaning of the symbols.

Proposition F.3. *For all ϵ small enough,*

$$|\lambda^* - \lambda_\epsilon| \leq ce^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}, \quad \|u^* - u_\epsilon\| \leq c_{(F.15)} e^{-a_{(F.15)}\epsilon^{-1}(1-x_0)} \quad (F.15)$$

Proof. Since $p_\epsilon = p^*$ in $\epsilon^{-1}[-1, 1]$, $\langle u_\epsilon^2 \rangle_* = \langle u_\epsilon^2 \rangle_\epsilon = 1$ so that $\lambda^* \geq \int u_\epsilon J^{\text{neum},*} * u_\epsilon$ and, by (D.11),

$$\|(J^{\text{neum},*} - J^{\text{neum},\epsilon}) * u_\epsilon\| \leq ce^{-a_{(\text{D.5})}\epsilon^{-1}(1-x_0)} \quad (\text{F.16})$$

where $J^{\text{neum},\epsilon}$ and $J^{\text{neum},*}$ are the kernel with Neumann conditions respectively in $\epsilon^{-1}[-1, 1]$ and $\epsilon^{-1}[-1, \ell^*]$. Thus

$$\lambda^* \geq \int u_\epsilon J^{\text{neum},\epsilon} * u_\epsilon - ce^{-a_{(\text{D.5})}\epsilon^{-1}(1-x_0)} \geq \lambda_\epsilon - ce^{-a_{(\text{D.5})}\epsilon^{-1}(1-x_0)} \quad (\text{F.17})$$

For the reverse inequality we write $\lambda_\epsilon \geq \frac{\int u^* J^{\text{neum},\epsilon} u^*}{\langle (u^*)^2 \rangle_\epsilon}$, the integral being extended to $\epsilon^{-1}[-1, 1]$. Using (F.16) we replace the kernel $J^{\text{neum},\epsilon}$ with $J^{\text{neum},*}$ and then extend the integral to $\epsilon^{-1}[-1, \ell^*]$ bounding u^* via (D.11) which holds as well for u^* in the whole $\epsilon^{-1}[-1, \ell^*]$ (see the paragraph “The operator A^* and its spectral properties” at the end of Appendix D). In this way we derive the first inequality in (F.15).

As in the proof of Proposition F.2 we bound $|u^*(x) - u_\epsilon(x)| \leq c'e^{-a_{(\text{F.15})}\epsilon^{-1}}$ when $|x - \epsilon^{-1}x_0| > r_{(\text{F.8})}\epsilon^{-1}$ using (D.11) (supposing $a_{(\text{F.15})} < a_{(\text{D.5})}r_{(\text{F.8})}$). When $|x - \epsilon^{-1}x_0| \leq r_{(\text{F.8})}\epsilon^{-1}$ we write

$$u^*(x) = (\lambda^*)^{-N} (A^*)^N u^*(x) = (\lambda^*)^{-N} A_\epsilon^N u^*(x) \quad (\text{F.18})$$

provided $(x_0 + r_{(\text{F.8})})\epsilon^{-1} + N \leq \epsilon^{-1}$, which is satisfied if $N = a\epsilon^{-1}$ with $a > 0$ small enough. Hence

$$u^*(x) = \left(\frac{\lambda_\epsilon}{\lambda^*}\right)^N \langle u^* u_\epsilon \rangle_\epsilon u_\epsilon(x) + A_\epsilon^N \tilde{u}^* \quad (\text{F.19})$$

$$|u^*(x) - \left(\frac{\lambda_\epsilon}{\lambda^*}\right)^N \langle u^* u_\epsilon \rangle_\epsilon u_\epsilon(x)| \leq ce^{-a_{(\text{D.34})}N} \quad (\text{F.20})$$

By (F.20) and since by (D.11) $\langle u^* \rangle_\epsilon \leq c$ and $|\langle (u^*)^2 \rangle_\epsilon - 1| \leq ce^{-a_{(\text{D.5})}\epsilon^{-1}(1-x_0)}$

$$|1 - \left(\frac{\lambda_\epsilon}{\lambda^*}\right)^N \langle u^* u_\epsilon \rangle_\epsilon^2| \leq ce^{-a_{(\text{D.34})}N} + ce^{-a_{(\text{D.5})}\epsilon^{-1}(1-x_0)} \quad (\text{F.21})$$

so that the second inequality in (F.15) follows from the first one. \square

As a corollary of Proposition F.2 and Proposition F.3 we have:

Corollary F.4. *In the same context of Proposition F.2,*

$$|\lambda^* - \lambda_{h,m}| \leq ce^{-a_{(\text{D.5})}\epsilon^{-1}(1-x_0)}, \quad \|u^* - u_{h,m}\| \leq c_{(\text{F.15})} e^{-a_{(\text{F.15})}\epsilon^{-1}(1-x_0)} \quad (\text{F.22})$$

APPENDIX G. CONVERGENCE OF THE ITERATIVE SCHEME

By (4.8) with $n = -1$ we have for $x \in \epsilon^{-1}[-1, 1]$,

$$\hat{h}_0(x) = -\epsilon j \int_{\epsilon^{-1}x_0}^x \chi(m_\epsilon(y))^{-1} = h^*(x) \quad (\text{G.1})$$

because $m_\epsilon = m^*$ on $\epsilon^{-1}[-1, 1]$ and (h^*, m^*) is a solution of (2.23) in $\epsilon^{-1}[-1, \ell^*]$. Thus by (4.8)

$$h_0(x) = h^*(x) - \frac{\int h^* u^*}{\int u^*} \quad (\text{G.2})$$

where the integrals are extended to $\epsilon^{-1}[-1, 1]$. Then, recalling (4.2),

$$h_0(x) - h_\epsilon(x) = -R_\epsilon(x) - \frac{\int h^* u^*}{\int u^*} \quad (\text{G.3})$$

Proposition G.1. *For all ϵ small enough $h_0 \in \mathcal{G}$ and*

$$N(h_0 - h_\epsilon) \leq c_{(\text{G.4})} \epsilon \quad (\text{G.4})$$

Proof. $\int_{\epsilon^{-1}(2x_0-1)}^{\epsilon^{-1}} h^* u^* = 0$ because h^* is antisymmetric and u^* symmetric around the middle point $\epsilon^{-1}x_0$ of the interval $\epsilon^{-1}[-1, \ell^*]$ (u^* is symmetric because the eigenvalue λ^* is simple and A^* symmetric). Since the estimates in Proposition D.1 apply to u^* as well (see the paragraph *The operator A^* and its spectral properties* at the end of Appendix D), by (D.11) and since $\|h^*\| \leq c$ we get

$$\int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^* h^* = \int_{-\epsilon^{-1}}^{\epsilon^{-1}(2x_0-1)} h^* u^* \leq c e^{-a_{(\text{D.5})} \epsilon^{-1}(1-x_0)} \quad (\text{G.5})$$

Recalling that $c_{(4.1)}$ in (4.1) is strictly positive uniformly in ϵ , we shall next prove that

$$\int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^* \geq \frac{c_{(4.1)}}{c_{(\text{D.11})}} \quad (\text{G.6})$$

By (D.11) $u^* \leq c_{(\text{D.11})}$, (G.6) then follows from (4.1):

$$1 = \langle (u^*)^2 \rangle_* = \int \frac{(u^*)^2}{p^*} \leq \int \frac{u^* c_{(\text{D.11})}}{c_{(4.1)}} = \left\{ \frac{c_{(\text{D.11})}}{c_{(4.1)}} \right\} \int u^*$$

Thus, recalling (4.6) and that $a_{(\text{D.5})}(1-x_0) > a_{(4.6)}$, see the paragraph *Fixing the parameters in the set \mathcal{G}* in Appendix F,

$$N\left(\frac{\int h^* u^*}{\int u^*}\right) = \sup_{|x| \leq \epsilon^{-1}} E_\epsilon(x) \frac{\int h^* u^*}{\int u^*} \leq c e^{-a_{(\text{G.7})} \epsilon^{-1}} \quad (\text{G.7})$$

with $0 < a_{(\text{G.7})} < a_{(\text{D.5})}(1-x_0) - a_{(4.6)}$. By Lemma 4.1, $N(R_\epsilon) \leq c\epsilon$ which together with (G.7) proves (G.4). Before proving that $h_0 \in \mathcal{G}$ we notice that by (D.11)

$$\int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^* |R_\epsilon| \leq c\epsilon e^{-a_{(\text{D.5})} \epsilon^{-1}(1-x_0)} \quad (\text{G.8})$$

a property which will be used in the sequel. We have already proved with (G.4) the first condition for $h_0 \in \mathcal{G}$. Then, since $\frac{d(h_0 - h_\epsilon)}{dx} = -\frac{dR_\epsilon}{dx}$ it will suffice to show that

$$\left| \frac{dR_\epsilon}{dx} \right| \leq c\epsilon^2 \mathbf{1}_{x \geq \epsilon^{-1}-1} \quad (\text{G.9})$$

We have

$$\frac{dR_\epsilon}{dx} = \int_{\epsilon^{-1}}^{\epsilon^{-1}+1} J(x, y) [\psi(y) - \psi(2\epsilon^{-1} - y)] dy, \quad \psi = \frac{dm^*}{dx} \quad (\text{G.10})$$

To bound the term $|\psi(x) - \psi(x + \xi)|$, $\xi = x' - x$, x and x' in $[\epsilon^{-1} - 1, \epsilon^{-1} + 1]$, in (G.10) we use the expression (D.41) for ψ . By (D.34)

$$\|L^{-1}\phi + \sum_{n=0}^N (A^*)^n \phi\| \leq c' \|\phi\| e^{-a(\text{D.34})N} \leq c'' \epsilon^3, \quad \phi = (\epsilon j) - (\epsilon j \int u^*) u^* \quad (\text{G.11})$$

if $N = C \log \epsilon^{-1}$ with C large enough. We have $(A^*)^n \phi = (A^*)^n (\epsilon j) - (\epsilon j \int u^*) (A^*)^n u^*$ By (D.11) and since $\lambda^* \in (0, 1)$

$$(A^*)^n u^*(x) \leq u^*(x) \leq c_{(\text{D.11})} e^{-a(\text{D.5})|x - \epsilon^{-1}x_0|}$$

so that $|\sum_{n=0}^N \{(A^*)^n \phi - (A^*)^n (\epsilon j)\}| \leq c\epsilon^3$ for $x \geq \epsilon^{-1} - 1$ and ϵ small enough. (G.9) will then follow from

$$\sum_{n=0}^N \left| \int (A^*)^n(x, y) (\epsilon j) dy - \int (A^*)^n(x', y) (\epsilon j) dy \right| \leq c\epsilon^2 \quad (\text{G.12})$$

x and x' in $[\epsilon^{-1} - 1, \epsilon^{-1} + 1]$. To prove (G.12) we write $\xi = x' - x$ and

$$\begin{aligned} \left| \int (A^*)^n(x, y) - \int (A^*)^n(x', y) \right| &\leq \left| \int A^*(x, x_1) \cdots A^*(x_{n-1}, x_n) \left| 1 - \prod \frac{p^*(x_i + \xi)}{p^*(x_i)} \right| \right| \\ &\leq c n \epsilon b^n, \quad b < 1 \end{aligned}$$

as all points above are in $\{x : x - \epsilon^{-1}x_0 > \epsilon^{-1} - (N + 1)\}$ (as $n \leq N$) and in such a region $0 < c_{(4.1)} < p^* < b < 1$ (as $m^* > m_\beta$) and $|p^*(x_i + \xi) - p^*(x_i)| \leq c\epsilon$, by Lemma 4.1. (G.12) is thus proved. \square

Proposition G.2. *There are $c_{(\text{G.13})}$ and $c_{(\text{G.14})}$ so that for all ϵ small enough the following holds. Suppose that for $n \geq 1$, both h_n and h_{n-1} are in \mathcal{G} , then*

$$N(m_n - m_{n-1}) \leq c_{(\text{G.13})} N(h_n - h_{n-1}) \quad (\text{G.13})$$

where $m_i = \tanh\{\beta J^{\text{neum}} * m_i + \beta h_i\}$, $i = n - 1, n$. Moreover

$$N(m_0 - m_\epsilon) \leq c_{(\text{G.14})} \epsilon \quad (\text{G.14})$$

Proof. We first prove (G.13) where we recall that $n \geq 1$. Let $t \in [0, 1]$ and $h(t) = th_n + (1 - t)h_{n-1}$. Since h_n and h_{n-1} are in \mathcal{G} then, by convexity, $h(t) \in \mathcal{G}$ and by Proposition F.1 there is $m(t)$ such that $(h(t), m(t)) \in \mathcal{A}$, in particular $m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$ and $m(0) = m_{n-1}$, $m(1) = m_n$ so that $|m_n - m_{n-1}| \leq \sup_{t \in [0, 1]} \left| \frac{dm(t)}{dt} \right|$. By (E.2), (D.11) and (D.35), recalling that $p_t \leq \beta$ and writing

$$\psi(x) := \left(\left| \int u(t) [h_n - h_{n-1}] \right| \right) e^{-a(\text{D.5})|x - \epsilon^{-1}x_0|} \quad (\text{G.15})$$

$$\left| \frac{dm(t)}{dt} \right| \leq c\epsilon^{-1}\psi + c \int e^{-a(\text{D.35})|x-y|} (|h_n - h_{n-1}|(y) + \psi(y)) dy \quad (\text{G.16})$$

We are going to prove that

$$\psi \leq c_{(G.17)} \epsilon^{10} e^{-a_{(D.5)} |x - \epsilon^{-1} x_0|} N(h_n - h_{n-1}) \quad (G.17)$$

By the definition of \mathcal{G} , $\int u^* h_i = 0$, $i = n-1, n$, then

$$\int u(t)[h_n - h_{n-1}] = \int [u(t) - u^*][h_n - h_{n-1}] \quad (G.18)$$

$$|\int [u(t) - u^*][h_n - h_{n-1}]| \leq N(h_n - h_{n-1}) \int |u(t) - u^*| E_\epsilon^{-1} \leq c \epsilon^{10} N(h_n - h_{n-1})$$

(by Corollary F.4). (G.17) is proved. Using (G.17) we have

$$\int e^{-a_{(D.35)} |x-y|} \psi(y) dy \leq c \epsilon^{10} e^{-a |x - \epsilon^{-1} x_0|} N(h_n - h_{n-1}), \quad a = \min\{a_{(D.5)}, a_{(D.35)}\}$$

The other integral on the r.h.s. of (G.16) is bounded by

$$\begin{aligned} \int e^{-a_{(D.35)} |x-y|} |h_n - h_{n-1}|(y) dy &\leq N(h_n - h_{n-1}) \int e^{-a_{(D.35)} |x-y|} E_\epsilon(y) dy \\ &\leq c e^{-a_{(4.6)} |x - \epsilon^{-1} x_0|} \end{aligned}$$

because $a_{(D.35)} > a_{(4.6)}$. Collecting all these bounds we have from (G.16)

$$\left| \frac{dm(t)}{dt} \right|(x) \leq c \left(\epsilon^9 e^{-a_{(4.6)} |x - \epsilon^{-1} x_0|} + (1 + \epsilon^{10}) e^{-a_{(4.6)} |x - \epsilon^{-1} x_0|} \right) N(h_n - h_{n-1})$$

which proves (G.13).

The proof of (G.14) goes in the same way except for (G.18) which becomes

$$\int u(t)[h_0 - h_\epsilon] = \int [u(t) - u^*][h_0 - h_\epsilon] + \int u^* h_\epsilon \quad (G.19)$$

By (G.5) and (G.8) the latter integral is bounded by $\leq c e^{-a_{(D.5)} \epsilon^{-1} (1-x_0)}$ and the bound (G.14) is not affected. \square

Proposition G.3. *There is $c_{(G.20)} \geq c_{(G.4)}$ so that for all ϵ small enough the following holds. Given any $n \geq 0$ if h_k , $k \leq n$, is well defined and in \mathcal{G} then also h_{n+1} is well defined and*

$$N(h_{k+1} - h_k) \leq \begin{cases} c_{(G.20)} \epsilon N(h_k - h_{k-1}), & k = 1, \dots, n \\ c_{(G.20)} \epsilon, & k = 0 \end{cases} \quad (G.20)$$

Proof. By Proposition F.1 there is m_k , $0 \leq k \leq n$, so that $(h_k, m_k) \in \mathcal{A}$ and, by (D.8), $p_k \equiv p_{h_k, m_k} \geq C_{(D.8)}$. As a consequence p_n^{-1} is bounded and h_{n+1} is well defined; moreover $|p_k^{-1} - p_{k-1}^{-1}| \leq c |m_k - m_{k-1}|$ and (for $x > \epsilon^{-1} x_0$)

$$|h_{k+1} - h_k| \leq c \epsilon \left(f + \frac{\int u^* f}{\int u^*} \right), \quad f(x) = \int_{\epsilon^{-1} x_0}^x |m_k - m_{k-1}| dy$$

Let $x > \epsilon^{-1} x_0$, then by (G.13) for $k \geq 1$

$$e^{a_{(4.6)} (\epsilon^{-1} - x)} f(x) = \left\{ \int_{\epsilon^{-1} x_0}^x e^{-a_{(4.6)} (x-y)} c_{(G.13)} \right\} N(h_k - h_{k-1}) \leq c N(h_k - h_{k-1}) \quad (G.21)$$

and by (D.11)

$$\begin{aligned} \int_{\epsilon^{-1}x_0}^{\epsilon^{-1}} u^* f &\leq cN(h_k - h_{k-1}) \int_{\epsilon^{-1}x_0}^{\epsilon^{-1}} e^{-a_{(4.6)}(\epsilon^{-1}-x)} e^{-a_{(D.5)}|x-\epsilon^{-1}x_0|} \\ &\leq cN(h_k - h_{k-1}) e^{-a_{(4.6)}\epsilon^{-1}(1-x_0)} \end{aligned} \quad (\text{G.22})$$

By (G.6) $\int u^*$ is bounded away from 0 hence the bound in (G.20) for $k > 0$ and $x \geq \epsilon^{-1}x_0$. When $k = 0$ we use (G.14) after bounding $|m_0 - m_\epsilon| \leq N(m_0 - m_\epsilon)E_\epsilon^{-1}$. Analogous bounds hold for $x < \epsilon^{-1}x_0$ and (G.20) is proved. \square

Proposition G.4. *In the same context of Proposition G.3, for any $k \leq n+1$*

$$N(m_k - m_\epsilon) \leq c\epsilon, \quad N(h_k - h_\epsilon) \leq c'\epsilon \quad (\text{G.23})$$

where $c = c_{(G.14)} + \frac{c_{(G.13)}c_{(G.20)}}{1 - \epsilon c_{(G.20)}}$, $c' = c_{(G.20)}(1 + \frac{1}{1 - \epsilon c_{(G.20)}})$. Moreover

$$N\left(\frac{d(h_k - h_\epsilon)}{dx}\right) \leq c\epsilon^2 \quad (\text{G.24})$$

and, in particular, $h_{n+1} \in \mathcal{G}$.

Proof. By (G.20) for $i \geq 0$, $N(h_{i+1} - h_i) \leq (\epsilon c_{(G.20)})^{i+1}$ and by (G.4), $N(h_0 - h_{-1}) \leq \epsilon c_{(G.4)} \leq \epsilon c_{(G.20)}$, $h_{-1} = h_\epsilon$. Then

$$N(h_k - h_\epsilon) \leq \sum_{i=0}^k N(h_i - h_{i-1}) \leq \epsilon c_{(G.20)} \left(1 + \frac{1}{1 - \epsilon c_{(G.20)}}\right)$$

hence the statement in (G.23) about h_k . The one about m_k is proved similarly, using (G.14) and (G.13). To prove (G.24) we write

$$\left|\frac{d(h_k - h_{k-1})}{dx}\right| \leq c\epsilon|m_{k-1} - m_{k-2}| \leq c'\epsilon|h_{k-1} - h_{k-2}| \quad (\text{G.25})$$

so that by (G.9), (G.20) and (G.4) and with $h_{-1} := h_\epsilon$, for $x > \epsilon^{-1}x_0$,

$$e^{a_{(4.6)}(\epsilon^{-1}-x)} \left|\frac{d(h_k - h_\epsilon)}{dx}\right| \leq e^{a_{(4.6)}(\epsilon^{-1}-x)} \left|\frac{d(h_0 - h_\epsilon)}{dx}\right| + c'\epsilon \sum_{i=1}^{k-1} N(|h_i - h_{i-1}|) \leq c''\epsilon^2$$

An analogous bound holds for $x < \epsilon^{-1}x_0$ hence (G.24). \square

Conclusion of the proof of Theorem 2.2. We shall first prove by induction that $h_n \in \mathcal{G}$ for all n . Indeed $h_0 \in \mathcal{G}$ by Proposition G.1 and by Proposition G.3 if $h_k \in \mathcal{G}$ for all $k \leq n$, then $h_{n+1} \in \mathcal{G}$. Thus $h_n \in \mathcal{G}$ for all n and by Proposition F.1 there is m_n so that $(h_n, m_n) \in \mathcal{A}$. We shall next prove that (h_n, m_n) converges in sup norm to a limit (h, m) and that, writing $h_{-1} = h_\epsilon$ and $m_{-1} = m_\epsilon$,

$$h = h_\epsilon + \sum_{n=0}^{\infty} (h_n - h_{n-1}), \quad m = m_\epsilon + \sum_{n=0}^{\infty} (m_n - m_{n-1})$$

The first series in fact converges because $N(h_{n+1} - h_n) \leq (c_{(G.20)}\epsilon)^{n+1}$, as remarked in the proof of Proposition G.4. The series for m converges for the same reason

because $N(m_n - m_{n-1}) \leq c_{(G.13)} N(h_n - h_{n-1})$. By (G.23), $N(m - m_\epsilon) \leq c\epsilon$ and $N(h - h_\epsilon) \leq c\epsilon$; moreover

$$m = \lim_{n \rightarrow \infty} m_n = \lim_{n \rightarrow \infty} \tanh\{\beta J^{\text{neum}} * m_n + \beta h_n\} = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$$

$$h = \hat{h} - \frac{\int \hat{h} u^*}{\int u^*}, \quad \hat{h}(x) = \int_{\epsilon^{-1}x_0}^x \frac{-\epsilon j}{\chi(m)}$$

because $h = \lim_{n \rightarrow \infty} \{\hat{h}_n - \frac{\int \hat{h}_n u^*}{\int u^*}\}$, $\hat{h}_n(x) = \int_{\epsilon^{-1}x_0}^x \frac{-\epsilon j}{\chi(m_n)}$. As a consequence, for any $z \in \epsilon^{-1}(-1, 1)$,

$$h(x) = h(z) + \int_z^x \frac{-\epsilon j}{\chi(m)}$$

so that the proof of Theorem 2.2 will be complete once we show that:

- there is x_ϵ such that $h(x_\epsilon) = 0$
- $\lim_{\epsilon \rightarrow 0} \epsilon x_\epsilon = x_0$

The existence of x_ϵ is proved using the implicit function theorem. We thus want to prove that $h(\epsilon^{-1}x_0)$ is “small”. Since $\hat{h}(\epsilon^{-1}x_0) = 0$ we need to control $|\int \hat{h} u^*|$. We write $\int \hat{h} u^* = \int (\hat{h} - \hat{h}_n) u^* + \int (\hat{h}_n - \hat{h}_0) u^* + \int \hat{h}_0 u^*$. The first term vanishes as $n \rightarrow \infty$ because $\int u^* < \infty$ and for any $x > \epsilon^{-1}x_0$ (for instance)

$$\begin{aligned} |\hat{h}(x) - \hat{h}_n(x)| &\leq |\epsilon j| \int_{\epsilon^{-1}x_0}^x |\chi(m)^{-1} - \chi(m_{n-1})^{-1}| \leq c\epsilon |x - \epsilon^{-1}x_0| \|m - m_{n-1}\| \\ &\leq c' \|m - m_{n-1}\| \rightarrow 0 \end{aligned}$$

having used that $\chi(m_n) = p_{h_n, m_n} \geq C_{(D.8)}$ and therefore $\chi(m) \geq C_{(D.8)}$ as $m_n \rightarrow m$ in sup norm. Analogously $|\chi(m_{n-1})^{-1} - \chi(m_\epsilon)^{-1}| \leq c|m_{n-1} - m_\epsilon| \leq cN(m_{n-1} - m_\epsilon)E_\epsilon^{-1} \leq c'E_\epsilon^{-1}$, so that for $x > \epsilon^{-1}x_0$

$$\begin{aligned} |\hat{h}_n(x) - \hat{h}_0(x)| &\leq |\epsilon j| \int_{\epsilon^{-1}x_0}^x |\chi(m_{n-1})^{-1} - \chi(m_\epsilon)^{-1}| \leq c\epsilon^2 \int_{\epsilon^{-1}x_0}^x E_\epsilon(y)^{-1} \\ &\leq c\epsilon^2 |x - \epsilon^{-1}x_0| e^{-a_{(4.6)} \epsilon^{-1}(1-x)} \end{aligned}$$

Thus by (D.11) the second term is bounded by $\int |\hat{h}_n - \hat{h}_0| u^* \leq c\epsilon^2 e^{-a_{(4.6)}(1-x_0)\epsilon^{-1}}$.

Finally, since by (G.1) $\hat{h}_0 = h^*$, by (G.5) $\int_{-\epsilon^{-1}}^{\epsilon^{-1}} \hat{h}_0 u^* \leq c_0 e^{-a_{(D.5)} \epsilon^{-1}(1-x_0)}$. In conclusion, letting $n \rightarrow \infty$,

$$|\int u^* \hat{h}| \leq c\epsilon^2 e^{-a_{(4.6)}(1-x_0)\epsilon^{-1}}, \quad |h(\epsilon^{-1}x_0)| \leq c'\epsilon^2 e^{-a_{(4.6)}(1-x_0)\epsilon^{-1}}$$

We shall next prove that h is continuous and that it changes sign in a small interval around $\epsilon^{-1}x_0$, thus concluding that there is x_ϵ in such interval where h vanishes. We have $\frac{dh}{dx}(x) = \frac{-\epsilon j}{\beta(1 - m^2(x))}$ which, by (D.8), is bounded. Moreover $N(m - m_\epsilon) \leq c\epsilon$ and $|m_\epsilon - \bar{m}_{x_0}| \leq c\epsilon$ in $[\epsilon^{-1}x_0 - 1, \epsilon^{-1}x_0 + 1]$. In such interval therefore $|\frac{dh}{dx}(x)| \geq a\epsilon$, $a > 0$. Hence there is x_ϵ where $h(x_\epsilon) = 0$ and

$$|x_\epsilon - \epsilon^{-1}x_0| \leq c''\epsilon e^{-a_{(4.6)}(1-x_0)\epsilon^{-1}} \quad (\text{G.26})$$

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